Strategic Decision Models

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Introduction to Games and their Theory

Representation of games: extensive and normal form, coalition and partition function form

Non cooperative Models
• Static games with complete information: Nash Equilibrium (NE, 1950)
• Dynamic games with complete information: Backward Induction, Subgame Perfect Nash Equilibrium (SPNE, Selten, 1965)

Cooperative Models
• Cost/Benefit cooperative games (von Neumann and Morgenstern, 1974)
• Set solutions: the imputation set and the core (Gillies, 1953)
• Point solutions: proportional allocations (Moulin, 1988), the Shapley value (Shapley, 1953), the Nucleolus (Schmeidler, 1969)

Biform Models: game theory applied to business strategy

Summary, Key Terms, and Problems
Recommended and Additional Bibliography
1. INTRODUCTION

1.1. Strategic decision models: game theory. Individual decision situations vs. interactive decision situations

1.2. Stepping stones in the history of game theory

1.3. Modeling a strategic decision situation: basic terminology and examples

1.4. Representation of games

1.4.1. Extensive form models

1.4.2. Normal form models

1.4.3. Coalition function form models

What is a game?

• Ordinary english: *pastime or diversion*

• Different varieties of games
  • Board games: Chess, Monopoly, ...
  • Card games: Solitaire, Poker, Mus, ...
  • Video games (played on a video screen against a computer)
  • Field games: football, basketball, hockey, ...

• Features in common
  • all games have rules
  • in every game strategic matters
  • there is an outcome to the game
  • this outcome depends on the strategies chosen by each of the players: *strategic interdependence*
We combine these features to define a **GAME** as

*any rule-governed situation with a well-defined outcome, characterized by strategic interdependence*

- Market game: firms competing in the same business
- *The outcome for a given firm depends not just on what strategy it chooses, but also on what strategies its competitors choose*
- A professional sports franchise, the Chicago Bulls (NBA): a corporation owned by private investors, that is in business to make money each year
- Coca-Cola Corporation (Atlanta, 1886). Coke and Pepsi have battled head-to-head for cola business: the cola wars (J.C. Louis, 1980)
- Economic negotiations
  - The White House and Congress on domestic economic policy issues
  - International economic negotiations among G-8 countries (trade and finance)
  - Various form of bargaining, negotiation, and arbitration: all games with at least an element of cooperation
What is game theory, and Why?

• Game theory (GT) is a study of strategic decision making: the study of \textit{mathematical models of conflict and cooperation between intelligent rational decision-makers} (Myerson, 1991)
  • \textbf{Intelligent}: has perfect knowledge of the game and of his opposition
  • \textbf{Rational}: makes decisions consistently in pursuit of his own objectives (maximize the expected utility payoff).

• We use GT to solve games, to help players avoid mistakes and find the right strategy if there is only one: sometimes the best we can do is narrow a player’s choice down to a set of strategies.

• \textit{A solution of a game should tell each player what outcome to expect and how to achieve that outcome}. This involve some mathematics [J. von Neumann & Oskar Morgenstern (1944); J. Nash (1950)].

• GT began as applied mathematics, it has become a \textbf{powerful mode of reasoning in business and economics}, and it is heavily used in other fields like formal \textit{political theory, evolutionary biology, psychology, and philosophy}. 
1.1. Strategic decision models: game theory

There are several reasons for studying GT, above and beyond the desire to know the truth that this science has to offer:

1. GT can **improve your strategic decision making:** it makes you more aware of when you are in a situation strategy matters, to say nothing of making you aware of strategy nuance on the part of your competitors or opponents.

2. It can **improve your ability to run a business** and to evaluate changes in policy: “competitive advantage”, “everyday low pricing”, and “winner’s curse” will make a lot of more sense to you after they have been strategically explicated.

3. GT can **help you become a better economist or a better manager:** it is the central paradigm of economics and finance (current buzzwords: market failure, credibility, incentive contracts, hostile takeovers, coalition building, …)

4. It might even make you a better Blackjack player (if you are into that sort of thing!)
**Individual decision situations**: every agent tries to maximize her expected utility payoff, regardless of the actions of its opponent

- *Decision Theory*, theory of one person games, or a game of a single player against nature
- *Consumer Theory*
- *Monopolistic Market* (prices, quantities)

**Interactive decision situations**: the outcome for a given agent depends not just on what strategy it chooses, but also on what strategies its competitors choose

- Strategic decision models, theory of n-person games (*game theory*)
- *Duopoly, Oligopoly*
- *Auctions*
- *Advertising investment by tobacco (car) companies*
J. Von Neumann and O. Morgenstern (1944)
Theory of Games and Economic Behavior
(the seminal work of game theory)

- Zermelo (1913): parlor games (with no randomness, like Chess).
- Borel (1921): 2-player zero-sum games
  - Pure and mixed strategies
  - Minimax theorem for special cases: existence of mixed-strategy equilibria in 2-player zero-sum games
- John von Neumann (1928): proved the minimax theorem
- von Neumann & Morgenstern (1944): method for finding mutually consistent solutions for two-person zero-sum games; cooperative games of several players; axiomatic theory of expected utility, which allowed to treat decision-making under uncertainty (second edition of this book)
- Cournot (1838): first important precursor of game theory and his analysis of duopoly (very close to Nash equilibrium for a duopoly model)
1.2. Stepping stones in the history of game theory


- The first mathematical discussion of the prisoner's dilemma (2-player non zero-sum game) appeared, and an experiment was undertaken by notable mathematicians Merrill M. Flood and Melvin Dresher, as part of the RAND corporation's investigations into game theory
  - It is formalized by A.W. Tucker (1955).

- John Nash (1950/51) developed a criterion for mutual consistency of players' strategies (Nash equilibrium), applicable to a wider variety of games than the criterion proposed by von Neumann and Morgenstern. In addition, he proposes the Nash Bargaining Solutions for Bargaining cooperative theory.

- Lloyd Shapley (1953) introduces a point solution (value) for cooperative games, and co-introduces the Core (set solution, Edgeworth, 1881) with D.B. Gillies (1959).

- R. Luce and H. Raiffa (1957): Games and Decisions.
**1960-1970**: geographic expansion across borders and Princeton (research centers in Israel, Germany, Belgium and and the former Soviet Union). *Strong connection* that emerges among *game theory*, *mathematical economics*, and *economic theory*.

- **Harsanyi** (1967, 1968) (*Games with incomplete information: Bayesian games*)

**1970-1986**: again *game theory* begins to awaken interest among economists and mathematicians.


- Applications of *game theory* to biology, computer sciences, moral philosophy, and cost allocation.

Since 1986 to date: has been significant development, with a considerable number of applications to social and economic sciences.

- **Economic Nobel Laureates (since 1969)**
- **1994**: John Nash, Reihnard Selten y John Harsanyi;
- **1996**: William Vickrey and James A. Mirrlees;
- **2005**: R. J. Aumann and C. Schelling;
- **2007**: Leonid Hurwicz, Eric S. Maskin y Roger B. Myerson.
- **2012**: Lloyd Shapley and Alvin Roth


Two basic approaches (von Neumann y Morgenstern, 1947)

- **Cooperative Models:** the players are able to form binding commitments. For instance, the legal system requires them to adhere to their promises.
  - Deals with coalitions and allocations, and considers group of players willing to allocate the joint benefits derived from their cooperation (however it takes place).

- **Non Cooperative Models:** no binding commitments are possible.
  - Deals with strategies and payoffs, and considers players willing to use strategies that maximize their individual payoffs.

- **Hybrid (Biform) games** (Brandenburger and Stuart, 2007) contain cooperative and non-cooperative elements: coalitions of players are formed in a cooperative game, but these play in a non-cooperative fashion.
Non cooperative and cooperative models

van Damme and Furth (2002)

“... non cooperative models assume that all the possibilities for cooperation have been included as formal moves in the game, while cooperative models are “incomplete” and allow players to act outside of the detailed rules that have been specified.”

Serrano (2008)

“One clear advantage of the (non cooperative) approach is that it is able to model how specific details of the interaction may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to those details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt.”
Non cooperative Models  
no binding commitments

- **Static games**: all players act simultaneously or, at least, without knowing the actions of the other (representation in normal form).

- **Dynamic games**: players have some information about the choices of other players (usually presented in extensive form).

- **Complete information**: each player knows all the information about the game. It is as if there was a “blue book” for every player, that any of them could look up.

- **Incomplete information**: same important feature of the game is unknown to some of the players: a player might lack some information about the set of players, the set of actions, or the payoff functions.
Cooperative Models
binding agreements between the players are possible

• With Transferible Utility (TU): all transfers of utility across players are assumed to be possible

  • Implicitly assumes that there is a *numéraire* good (for instance, money) such that the utilities of all players are linear with respect to it and that this good can be freely transferred among players.

• Without Transferible Utility (NTU): utility is NOT transferable among players

  • Even if money is available in large amounts (which is not always the case), we find that the several players’ utility for money is not linear, and not always independent of their other assets.

• *TU-games* (described by a number for each coalition of players) are much more tractable than general *NTU-games* (a set for each coalition): **games with and without side payments (TU and NTU, respectively)**.
1.3. Basic terminology

• **Players**: decision-makers focused on maximizing their expected utility payoff.

• **Actions of each player**: decisions (moves) to be made by each player at the time to play (finite or infinite set).

• **Strategies**: comprehensive plan of actions available to each player
  - **Strategy profiles**: a combination of strategies that all the players might choose.

• **Outcomes to the game**: different ways to complete the game (every outcome always has consequences)

• **Payoffs**: each player utility ascribed to a game outcome (valuation for the player with the consequences of achieving a particular outcome).
Example (Extensive Form)

Auction of a 50 euros note

Peter auction a bill of 50 euros between Carlos and Blanca according to the following rules: is played in turns. The one who gets to play with could pass or bid 20 euros more than the previous (assuming that has). Blanca starts (passing or betting 20 euros). If a player decides to pass, it can not bid on any play black. Last bidder wins, who gets the bill. If none has bid, take 25 euros each. Both players must pay their last bid. Apart from the rules, t is common knowledge that each player has 60 euros.

- **Player set**: \{Blanca, Carlos\}=\{1,2\}

- **Strategies**: pass (P) or bid 20 euros more than the previous (20, 40,60)

- **Payoffs (benefit for each player)**: (25,25), (0,30), (30,0), (-20,10), (-10,-40)
Example (Extensive Form)

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Example (Normal Form)

Matching euros

Ana (1) and Robert (2) *simultaneously* deposited two one euro coins on a table. If there are two heads or two tails, Ana collects two euros, while if there is a face (C) and a cross (X), Robert takes the two euros.
Example (Coalition Function Form)

Böhm-Bawerk horse market  (Böhm-Bawerk, 1923; Shubik, 1987)

A farmer has a cow that can sell in the market at a profit of one unit. To sell the cow is imperative that it passes the farm of one of its two neighbors.

- **Player set:** \{neighbor 1, neighbor 2, owner\}={1, 2, 3}
- **Coalitions:** \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}
- **Joint profits:** 0, 0, 0, 0, 1, 1, 1

How to allocate the joint profits?

(Solutions cooperative models)
1.4. Representation of games

**Extensive Form and Normal (Strategic) Form (non cooperative)**

- Both of them specified players, actions and payoffs
- **Extensive** can be used to formalize games with a time sequencing of moves (games here are played on trees).

- **Normal (Strategic)** is focused on players’ strategies as if they were able to act simultaneously or, at least, without knowing the actions of the other (usually represented by a matrix).

- Every extensive-form game has an equivalent normal-form game, however the transformation to normal form may result in an exponential blowup in the size of the representation, making it computationally impractical (Leyton-Brown & Shoham, 2008).
Coalition Function Form and Partition Function Form (cooperative)

- **Coalition Function** deals with groups of players to coordinate their actions and pool their joint profits (benefits/costs) *(usually represented by a characteristic function)*.
  
  - N={1, 2, 3}
  - v({1})=v({2})=v({3})=v({1,2})=0
  - v({1,3})=v({2,3})=v({1,2,3})=1

- The coalition function form ignores the possible externalities of coalition formation.
  - In economics an externality is a cost or benefit that is not transmitted through prices in that it is incurred by a party who was not involved as either a buyer or seller of the goods or services causing the cost or benefit.

- **In the Partition Function form** the payoff of a coalition depends not only on its members, but also on the way the rest of the players are partitioned *(Thrall & Lucas, 1969)*.
1.4.1. Extensive Form models

Extensive Form models

- **Finite set of players:** $N = \{0, 1, 2, \ldots, n\}$; random or nature moves
- **Finite set of nodes:** $X$ (each node represents a possible game situation)
  
  $\sigma : X \to X$ / $\sigma(x)$ predecessor node $x$ immediately ($\sigma(0) = 0$)

  $T(x) = \{x \in X / S(x) = \sigma^{-1}(x) = \emptyset\}$ terminal nodes

  $D(x) = \{x \in X / S(x) \neq \emptyset\} = X - T(x)$ decision nodes

- **Set of all possible actions:** $A$

  $\alpha : X - \{0\} \to A$ / $\alpha(x)$ action from $\sigma(x)$ to $x$

  $x, x' \in s(x), x \neq x'$, $\alpha(x) \neq \alpha(x')$ actions based on the same node and lead to different nodes

  should be different

  $\forall x \in D(x), A(x) = \{a \in A / \exists x' \in s(x) \text{ con } a = \alpha(x')\}$ set of available actions from $x$

- **Set of nodes in which player $i$ has to choose an action:** $X_i, i \in N$

  $\bigcup_{i \in N} X_i = D(x), X_i \cap X_j = \emptyset, \forall i, j \in N, i \neq j$

  In a particular decision node moves only one player
• **Family of information sets:** $H$

$h : X \rightarrow H / h(x)$ information set which the node $x$ belongs to; partition of $D(x)$

All the nodes belonging to the same information set have available

the same actions: $h(x) = h(x') \Rightarrow A(x) = A(x')$

$h = h(x) \in H, A(h) = \{a \in A / a \in A(x) \text{ para } x \in h\}$ set of actions available

in the information set $h$

$H_i$ set of all information sets for player $i$

$H = \bigcup_{i \in N} H_i$ contains all the information sets included in the $H_i$

• **The probability assignment**

\[
\rho : H_0 x A \rightarrow [0,1] / \rho(h, a) = 0 \text{ si } a \notin A(h) \text{ y } \sum_{a \in A(h)} \rho(h, a) = 1, \forall h \in H_0
\]

• **The payoffs:** payoff/utility received by each player if it has reached the terminal node $x$

\[
\pi : T(x) \rightarrow R^n / \pi(x) = (\pi_1(x), ..., \pi_n(x))
\]
1.4.1. Extensive Form models

**Extensive games**

\[ \Gamma = \{N,(X,\sigma),(A,\alpha),\{X\}_{i \in N},\{H\}_{i \in N},(A(h))_{h \in H},\rho,\pi\} \]

- Player set \( N \)
- Set of nodes/predecessor function \( (X,\sigma) \)
- Set of all possible actions \( (A,\alpha) \)

- Family of nodes in which every player has to choose an action \( \{X\}_{i \in N} \)
- Family of information sets \( \{H\}_{i \in N}; H = \bigcup_{i \in N} H_i \)

- Family action sets available at each information set \( \{A(h)\}_{h \in H} \)

- Probability assignment to actions in information sets with random moves \( \rho \)

- Payoff function \( \pi \)
Consider a Spanish pack with cards shuffled. Each player 1 and 2 deposits a 5 euros bill on the table. Player 1 takes a card and nobody else looks it at. He can bid 5 euros more on the table (c,e) or pass (d,f). If he passes, player 1 gets coin or a cup (a), and player 2 gets the money if the card is a sword or a club (b). On the contrary, if he bids, player 2 can accept the bid (g) depositing 5 euros more on all the money on the table if the card is a the table, or pass (h). In the first case, either player 1 gets all the money if the card is a coind or a cup (a), or player 2 does if the card is a sword or a club (b). If player 2 passes, player 1 gets all the money whatever the chosen card.

\[ N = \{0,1,2\}; X = \{o, x1, x2, x3, x4, x5, x6, x7, x8, x9, x10\} \]

\[ \sigma(o) = o, \sigma(x1) = \sigma(x2) = o, \sigma(x3) = \sigma(x4) = x1, \sigma(x5) = \sigma(x6) = x2, \sigma(x7) = \sigma(x8) = x3, \sigma(x9) = \sigma(x10) = x5 \]

\[ A = \{a, b, c, d, e, f, g, h\} \]

\[ \alpha(x1) = a, \alpha(x2) = b, \alpha(x3) = c, \alpha(x4) = d, \alpha(x5) = e, \alpha(x6) = f, \alpha(x7) = g \alpha(x8) = h, \alpha(x9) = g, \alpha(x10) = h \]

\[ X_0 = \{o\}, X_1 = \{x1, x2\}, X_2 = \{x3, x5\}; H_0 = \{\{o\}\}, H_1 = \{\{x1\}, \{x2\}\}, H_2 = \{\{x3, x5\}\}; H = \{\{o\}, \{x1\}, \{x2\}, \{x3, x5\}\} \]

\[ A(\{o\}) = \{a, b\}, A(\{x1\}) = \{c, d\}, A(\{x2\}) = \{e, f\}, A(\{x3, x5\}) = \{g, h\} \]

\[ \rho(\{o\}, a) = 1/2, \rho(\{o\}, b) = 1/2 \]

\[ \pi(x4) = (5, -5), \pi(x6) = (-5, 5), \pi(x7) = (10, -10), \pi(x8) = (5, -5), \pi(x9) = (-10, 10), \pi(x10) = (5, -5) \]
1.4.2. Strategic Form models

Strategic Form models

• **Player set**: \( N = \{1, 2, ..., n\} \)
• **Strategy**: contingency plan, complete or decision rule, for a player, which specifies how the player will act in every possible circumstance that corresponds to move (the set of such a circumstances corresponds to the family of information sets of the player)

  • **Strategy**: \( s_i : H_i \rightarrow A / s_i(h) \in A(h), i = 1, ..., n \)

  • **Set of strategies**: \( S_i, i = 1, ..., n \)

  • **Strategy profile or strategy combination**: \( s = (s_1, s_2, ..., s_n) \in S_1 \times S_2 \times ... \times S_n = S \)

  • **Payoff/utility function**: \( u(s) = (u_1(s), u_2(s), ..., u_n(s)) = (\pi_1(n(s)), \pi_2(n(s)), ..., \pi_n(n(s))) \)

    • \( n(s) \) terminal node reached if the players have chosen the combination \( s \)

Strategic Games

\[
G = \{N_i, (S_i)_{i \in N}, (u_i)_{i \in N}\}
\]
Example (strategic game)  
Auction of a 50 euros note

Strategies for player 1 (Blanca):
• : P, P if J1 20 and J2 40
• : P, 60 if J1 20 and J2 40
• : 20, P if J1 20 and J2 40
• : 20, 60 if J1 20 and J2 40

Strategies for player 2 (Carlos):
• : P si J1 P y P si J1 20
• : P si J1 P y 40 si J1 20
• : 20 si J1 P y P si J1 20
• : 20 si J1 P y 40 si J1 20

<table>
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<th>Blanca \ Carlos</th>
<th>$s_1^1$</th>
<th>$s_2^1$</th>
<th>$s_2^2$</th>
<th>$s_3^3$</th>
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<td>25, 25</td>
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<tr>
<td>$s_1^3$</td>
<td>30, 0</td>
<td>-20, 10</td>
<td>30, 0</td>
<td>-20, 10</td>
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<tr>
<td>$s_1^4$</td>
<td>30, 0</td>
<td>-10, -40</td>
<td>30, 0</td>
<td>-10, -40</td>
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</table>
Coalition Function Form models

• **Player set**: \( N = \{1, 2, \ldots, n\} \)

• **Possible coalitions in** \( N \): \( P(N) \) (including the empty coalition)

• **Transferible Utility**: benefits or costs generated by a coalition can be allocated among the players in the coalition

• **Characteristic function**: assigns to each coalition the minimum value that can be obtained if all its members are associated and act as a group

\[
\nu: P(N) \rightarrow \mathbb{R} / \nu(\emptyset) = 0, \nu(S)\forall S \subseteq N
\]

• **Cooperative TU game**: \((N, \nu)\)
Example (cooperative game) Producers of goods

Consider three companies that produce the same good. Given their technologies, company 1 can produce 0, 8 or 16 units of output with a unit cost of 2 currencies. Company 2 can produce 0, 4 or 12 units at the same unit cost, and the company 3 can produce 0, 8 or 12 units at the same unit cost. The inverse demand function is well known for the three companies and is given by

\[ p(x) = 35 - 0.75x / x \]  

total amount to hit the market

Strategic game

\[ \{\{1, 2, 3\}, (S_1, S_2, S_3), (u_1, u_2, u_3)\} \]

\[ S_1 = \{0, 8, 16\}, S_2 = \{0, 4, 12\}, S_3 = \{0, 8, 12\} \]

\[ x_1 \in S_1, x_2 \in S_2, x_3 \in S_3, x = x_1 + x_2 + x_3 \]  

total amount of product to hit the market

\[ u_i(x_1, x_2, x_3) = p(x)x_i - 2x_i, i = 1, 2, 3 \]  

benefit function for each company
### 1.4.3. Coalition Function Form models

#### Firm 3 vs Firm 2

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<thead>
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<th>Firm 2</th>
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<tr>
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#### Firm 3 vs Firm 2

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#### Firm 3 vs Firm 2

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<td>16</td>
<td>192,0,144</td>
<td>144,36,108</td>
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Cooperative game \((\{1,2,3\}, v)\)

<table>
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<th>(S)</th>
<th>(\emptyset)</th>
<th>{1}</th>
<th>{2}</th>
<th>{3}</th>
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<th>{1,3}</th>
<th>{2,3}</th>
<th>{1,2,3}</th>
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<td>72</td>
<td>36</td>
<td>48</td>
<td>192</td>
<td>192</td>
<td>144</td>
<td>360</td>
</tr>
</tbody>
</table>

**Payoff \{1,2\}**

<table>
<thead>
<tr>
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<th>(x_2)</th>
<th>Payoff {1,2}</th>
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<td>(\min{336,240,192}=192)</td>
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\(x_1 = 0 \rightarrow 0\)
\(x_1 = 8 \rightarrow \min\{216,192,144,168,96,120,72\} = 72\)
\(x_1 = 16 \rightarrow \min\{336,288,240,192,96,144,48\} = 48\)
\(\max\{0,72,48\} = 72 = v(\{1\})\)

**Firm 3**

<table>
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**Firm 1**

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**Firm 3**

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### Cooperative game \((\{1,2,3\}, v)\)

<table>
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<th>Ø</th>
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<th>{2}</th>
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<table>
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</table>
Summary 1

1. A **game** is any rule-governed situation with a well-defined outcome, characterized by strategic interdependence among the players.

2. **Game theory** is the science that studies mathematical models of conflict and cooperation between intelligent rational decision-makers.

3. We can say that **game theory began in the early 19th century** (von Neumann & Morgenstern, 1944) and since then it has been significant development, with a considerable number of applications to social and economic sciences. An example of this are the **five economic Nobel laureates** (1994, 1996, 2005, 2007, and 2012).

4. There are **two basic approaches** to game theory (von Neumann & Morgenstern, 1947): cooperative games (the players are able to form binding commitments), and non cooperative games (no binding commitments are possible).

5. There are **three representations** of games: extensive form, strategic form (both non cooperative games), and coalition function form (cooperative game).

6. The **extensive form** is the basic description of a game. An extensive form is a tree diagram, with nodes, branches, an initial node, information sets, and endpoints. The **strategic form** looks at the implications of strategies, while suppressing some of the detail contained in the extensive form. The **coalition function form** suppresses even more detail than the strategic form and is used mainly for studying cooperation among the players.
## Key Terms 1

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
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<tbody>
<tr>
<td>game</td>
<td>Strategic interdependence market game economic negotiation game theory decisions rational decision-maker individual/interactive decision situation non cooperative/cooperative model complete/incomplete information static/dinamic game TU game</td>
</tr>
<tr>
<td>player</td>
<td>strategy strategy profile payoff extensive form game strategic form game coalition function form actions nodes information set coalition characteristic function</td>
</tr>
</tbody>
</table>
1. Problems

Problems 1

1.1. Describe the elements of the following extensive form models: (1) Auction of a 50 euros note, (2) Coin game.

1.2. Consider a 2-person game in which both players want to get a pot of gold. Player 1 moves first, and can go either left or right. If player 1 goes right, the game ends and nobody gets anything. If player 1 goes left, then it is player 2’s turn to move. Player 2 can go either left or right. If player 2 goes right, the game ends and nobody gets anything. If player 2 goes left, then game ends as well. But now he players have found the pot of gold, whose value they divide evenly. Draw the extensive form and try to solve it. Transform the extensive form in the strategic form, and compare them.

1.3. Consider 2-person game with perfect information, which is called a game like chess. It satisfies the following requirements. First, the players have perfect information (they know exactly what has happened every time a decision needs to be made). Second, each player has at most a finite number of strategies. Third, the outcomes are limited to win, lose, or draw. Either player 1 wins and player 2 loses (w,l), or both players get a draw (d,d) or player 1 loses and player 2 wins (l,w). Draw the extensive form of a game like chess with 2 strategies for each player and try to solve it. Transform the extensive form in the strategic form, and compare them.

1.4. Transform the strategic form in the coalition function form for the 2-person pot of gold game, and the 2-person game like chess.
2. NON COOPERATIVE MODELS

2.1. Static games with complete information

2.1.1. Nash Equilibrium (Nash, 1950) and Pareto efficiency. Examples
2.1.2. Mixed strategies and mixed strategies NE in finite games. Computing mixed strategy NE

2.2. Dynamic games with complete information

2.2.1. Perfect and imperfect information
2.2.2. Examples: the iterated prisioner’s dilemma
2.2.3. Strategies in finite dynamic games: mixed strategies v.s. behavior strategies
2.2.4. Backward Induction: Subgame Perfect Nash Equilibrium (Selten, 1965)

Practical sessions

2.3. Quantity Competition among firms: Cournot Oligopoly (1838)
2.4. Price Competition between two firms: Bertrand Duopoly (1878)
2.5. Credible Quantity Competition: Stackelberg leadership model (1934)
Non Cooperative models

No binding commitments are possible: deals with strategies and payoffs, and considers players willing to use strategies that maximize their individual payoffs.

- **Static games** (finite: representation in strategic form)
- **Dynamic games** (finite: representation in extensive form).
- **Complete information** (each player knows all the information about the game; deterministic games)
- **Incomplete information** (same important feature of the game is unknown to some of the players; bayesian games).
Static games with complete information

• **Elements**: players, strategies available and payoffs for each player (utility for each outcome of the game).
• **“Games with simultaneous moves”**: all the players make their decisions simultaneously and independently, and they get the payoffs depending on the strategy profiles.
• It is **common knowledge** the complete structure of the game: all the players know the strategies or actions available for each of them and the payoffs for every strategy profile and, moreover, everyone knows that everyone knows, and everyone knows that everyone knows that everyone knows... and so on.

• **Strategic Games representation**

\[ G = \{N; S_1, S_2, \ldots, S_n; u_1, u_2, \ldots, u_n\} \]

\[ s = (s_1, s_2, \ldots, s_n) \in S = S_1 \times S_2 \times \cdots \times S_n \text{ strategy profile} \]

\[ s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_{-i} = S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times S_n \]

\[ u_i(s_1, s_2, \ldots, s_n) \in R \]

• **G finite game**: finite number of players and/or finite strategy sets
Prisoner’s dilemma (M. Flood & M. Dresher, 1950; A. Tucker 1955)

Canonical example that shows why two individuals might not cooperate, even if it appears that it is in their best interests to do so.

Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of speaking to or exchanging messages with the other. The police admit they don't have enough evidence to convict the pair on the principal charge. They plan to sentence both to a year in prison on a lesser charge. Simultaneously, the police offer each prisoner a Faustian bargain. If he testifies against his partner, he will go free while the partner will get three years in prison on the main charge. Oh, yes, there is a catch ... If both prisoners testify against each other, both will be sentenced to two years in jail.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Not Defet</td>
<td>-1, -1</td>
<td>-3, 0</td>
</tr>
<tr>
<td>Defet</td>
<td>0, -3</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>
Battle of the sexes (a coordination game)

Imagine a couple that agreed to meet this evening, but cannot recall if they will be attending the opera or a football match (and the fact that they forgot is common knowledge). The husband (2) would most of all like to go to the football game. The wife (1) would like to go to the opera. Both would prefer to go to the same place rather than different ones. If they cannot communicate, where should they go?

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>3,2</td>
<td>0, 0</td>
</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

This representation does not account for the additional harm that might come from not only going to different locations, but going to the wrong one as well (e.g. he goes to the opera while she goes to the football game, satisfying neither). To account for this, the game is sometimes represented as in "Battle of the Sexes (2)".

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Football</th>
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</thead>
<tbody>
<tr>
<td>Opera</td>
<td>3,2</td>
<td>1, 1</td>
</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
<td>2, 3</td>
</tr>
</tbody>
</table>
Hawk-Dove Game (Chicken Game)

The principle of the game is that while each player prefers not to yield to the other, the worst possible outcome occurs when both players do not yield.

There is a competition for a shared resource and the contestants can choose either conciliation (Dove) or conflict (Hawk). The value for the resource is $V (>0)$. The contestants know that if both of them choose conciliation they share the resource; however, if both choose conflict, it takes each of them a cost of $C (>0)$. On the other hand, if everyone chooses a different behavior, the conciliator gets nothing while the conflict takes all.

<table>
<thead>
<tr>
<th></th>
<th>Dove</th>
<th>Hawk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dove</td>
<td>$V/2, V/2$</td>
<td>0, $V$</td>
</tr>
<tr>
<td>Hawk</td>
<td>$V, 0$</td>
<td>$V/2-C, V/2-C$</td>
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</table>
The Nash Demand Game (infinite game)

Two players split a cake according to the following rules: both write, simultaneously, a number between 0 and 1, which means the portion of the cake they are willing to get. If the sum of the two numbers is less than or equal to 1, each player receives the requested. Otherwise, none of them gets nothing.

\[
G = \{ N ; S_1 , S_2 ; u_1 , u_2 \} \\
N = \{ 1 , 2 \}; \quad S_1 = S_2 = [ 0 , 1 ] \\
u_1 ( s_1 , s_2 ) = \begin{cases} 
    s_1 & \text{if } s_1 + s_2 \leq 1 \\
    0 & \text{if } s_1 + s_2 > 1 
\end{cases} \\
u_2 ( s_1 , s_2 ) = \begin{cases} 
    s_2 & \text{if } s_1 + s_2 \leq 1 \\
    0 & \text{if } s_1 + s_2 > 1 
\end{cases}
\]
Nash Equilibrium (Nash, 1950)

Which properties should

• satisfy a strategy profile to be established as a solution of the strategic game?
• have a strategy profile for us to think that is a good prediction of the behavior of rational players?

• Nash Equilibrium (NE): a strategy profile such that no player gains when unilaterally deviating from it; i.e., NE concept searches for rest points of the interactive situation described by the strategic game.

\[ G = \{ N; S_1, ..., S_n ; u_1, ..., u_n \} \]

\[ s^* = (s^*_1, s^*_2, ..., s^*_i, ..., s^*_n) \] is NE if for each player \( i \),

\[ u_i (s^*_1, s^*_2, ..., s^*_i, ..., s^*_n) \geq u_i (s^*_{i-1}, s^*_2, ..., s^*_i, ..., s^*_n) \] for all \( s^*_i \in S_i \)

For every player \( i \), \( s^*_i \) is a solution for the following optimization problem

\[ \max u_i (s^*_1, s^*_2, ..., s^*_{i-1}, s^*_i, s^*_{i+1}, ..., s^*_n), s^*_i \in S_i \]

For every player \( i \), \( s^*_i \) is a best response to \( s^*_{-i} \)
• **Prisoner’s dilemma:** $\text{NE}(G) = \{(\text{Defet, Defet})\}$

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<tbody>
<tr>
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<td>-2, -2</td>
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• **Battle of the sexes:** $\text{EN}(G) = \{(\text{Opera, Opera}), (\text{Football, Football})\}$

<table>
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</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

• **Hawk-Dove Game:**
  - $\text{EN}(G) = \{(\text{Hawk, Hawk}), \text{if } V/2 > C\}$
  - $\text{EN}(G) = \{(\text{Hawk, Dove}), (\text{Dove, Hawk}), \text{if } V/2 < C\}$

<table>
<thead>
<tr>
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<tr>
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<tr>
<td>Hawk</td>
<td>$V$, 0</td>
<td>$V/2-C$, $V/2-C$</td>
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</tbody>
</table>
• **Matching euros**: \( EN(G) = \{\emptyset\} \)

<table>
<thead>
<tr>
<th>1 \ 2</th>
<th>C</th>
<th>X</th>
</tr>
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<td>1, -1</td>
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</table>

• **Auction of a 50 euros note**: \( EN(G) = \{(s_1^4, s_2^1), (s_1^4, s_2^3), (s_1^1, s_2^4), (s_1^2, s_2^4)\} \)

<table>
<thead>
<tr>
<th>Blanca\Carlos</th>
<th>( s_2^1 )</th>
<th>( s_2^2 )</th>
<th>( s_2^3 )</th>
<th>( s_2^4 )</th>
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<td>( s_1^2 )</td>
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<td>25, 25</td>
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<td>0, 30</td>
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<tr>
<td>( s_1^3 )</td>
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<td>30, 0</td>
<td>-20, 10</td>
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<tr>
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<td>-10, -40</td>
<td>30,0</td>
<td>-10, -40</td>
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2.1.1. Nash Equilibrium as Best Response

**Best Response Correspondence**

- Infinite games: computing NE requires solving several simultaneous optimization problems (one for each player).

- The best response is the strategy (or strategies) which produces the most favorable outcome for a player (optimum(a)), taking other players' strategies as given (Gibbons, 1992).

- Systematic search for NE: calculate the best response (or one of the best responses) to the other players' strategies.

$$G = \{N; S_1, ..., S_n; u_1, ..., u_n\}; \quad s_j = (s_1, s_2, ..., s_{j-1}, s_{j+1}, ..., s_n)$$

**Best Response correspondence of player i:**

$$R_i(s_{-i}) \text{ best response correspondence of player } i \text{ to } s_{-i}$$

$$s_i^* = R_i(s_{-i}) \text{ if and only if}$$

$$u_i(s_1, s_2, ..., s_{j-1}, s_{j+1}, ..., s_n) \geq u_i(s_1, s_2, ..., s_{i-1}, s_i^*, s_{i+1}, ..., s_n), \forall s_j \in S_j$$

- **NE characterization:** strategy profiles where all the player's best responses agree

$$G = \{N; S_1, ..., S_n; u_1, ..., u_n\};$$

$$s^* = (s_1^*, s_2^*, ..., s_i^*, ..., s_n^*) \text{ is NE if and only if } s_i^* = R_i(s_{-i}^*) \text{ for every player } i$$
2.1.1. Nash Equilibrium: examples

- The Nash Demand Game:

\[ EN(G) = \left\{ (s_1^*, s_2^*) \in S_1 \times S_2 \mid s_1^* + s_2^* = 1 \right\} \cup \{(1,1)\} \]

\[ G = \{ N ; S_1, S_2 ; u_1, u_2 \} \]

\[ N = \{1, 2\}; \quad S_1 = S_2 = [0, 1] \]

\[ u_1(s_1, s_2) = \begin{cases} s_1 & \text{if } s_1 + s_2 \leq 1 \\ 0 & \text{if } s_1 + s_2 > 1 \end{cases} \]

\[ u_2(s_1, s_2) = \begin{cases} s_2 & \text{if } s_1 + s_2 \leq 1 \\ 0 & \text{if } s_1 + s_2 > 1 \end{cases} \]

Player 1: (1) \[
\left\{ \begin{array}{l}
\max \ s_i \\
0 \leq s_i \leq 1 \\
s_i + s_2 \leq 1
\end{array} \right.
\]

- For (1) \[
R_1(s_2) = \begin{cases} 1 - s_2 & \text{if } s_2 < 1 \\
[0, 1] & \text{if } s_2 = 1 \end{cases}
\]

- For (1) \[
R_2(s_1) = \begin{cases} 1 - s_1 & \text{if } s_1 < 1 \\
[0, 1] & \text{if } s_1 = 1 \end{cases}
\]
2.1.1. Existence of pure strategy NE

Existence of Nash equilibria

A generalization of the original Nash Theorem (1950). This version was proved in Rosen (1965).

**Theorem (Nash theorem).** Let \( G=\{N; S_1, \ldots, S_n; u_1, \ldots, u_n\} \) be a strategic game such that, for each player \( i \),

1. \( S_i \) is a nonempty, convex, and compact subset of \( R^s \)
2. \( u_i \) is continuous in \( S \)
3. For each \( s_{-i} \), \( u_i(s_{-i}) \) is quasi-concave on \( S_i \)

Then, the game \( G \) has, at least, one Nash equilibrium.

**Quasi-concavity of the function** \( u_i(s_{-i}) \): provided that all players different from player \( i \) are playing according to \( s_{-i} \), we have that, for each real number \( K \), if two strategies give at least payoff \( K \) to player \( i \), then so does any convex combination of them.
2.1.1. Pareto efficiency

Pareto efficiency (V. Pareto, 1848-1923)

The Nash equilibrium may sometimes appear non-rational in a third-person perspective. This is because it may happen that a Nash equilibrium is not Pareto optimal.

A strategy profile is **Pareto efficient (optimal)** if no player can be made better off without making at least one individual worse off (i.e., it can not be changed by another profile without no player losing out and out gaining any strictly.

\[
G = \{N; S_1,..., S_n ; u_1,..., u_n \}
\]

\[
s = (s_1, s_2, ..., s_j, ..., s_n )\text{ is Pareto dominated by } s' = (s'_1, s'_2, ..., s'_j, ..., s'_n )\text{ if and only if } \]

\[
u_i (s') \geq u_i (s) \text{ for all } i = 1,..., n, \exists j \text{ such that } u_j (s') > u_j (s).
\]

\[s^* \text{ is Pareto optimal (efficient) if and only if it is not Pareto dominated by another profile: } \]

\[\forall s \in S, s \neq s^*, u_i (s^*) > u_i (s) \text{ for all } i \in N.\]

\[s^* \text{ is inefficient if it is dominated by some other profile.}\]

2.1.1. Pareto efficiency

- While Pareto dominance is a concept of social efficiency analysis, relevant to the group of players as a group, strategies domination, implicit in NE, is a concept of individual efficiency analysis, relevant to each player as an individual agent.

- Nash and Pareto optimality are independent concepts:
  - Prisioner’s Dilemma: (Defet, Defet) is inefficient since it is Pareto dominated by (Not Defet, Not Defet).
  - The Battle of the sexes: all NE are Pareto efficient,
  - The Nash Demand Game: all NE are Pareto efficient, but (1,1) is not.

\[ G = \{ N; S, u \} \]
\[ s = (s_1, s_2, ..., s_n) \text{ is Pareto dominated by } s' = (s'_1, s'_2, ..., s'_n) \text{ if and only if } \]
\[ u_i(s') \geq u_i(s) \text{ for all } i = 1, ..., n, \exists j \text{ such that } u_j(s) > u_j(s'). \]
\[ s^* \text{ is Pareto optimal if and only if it is not Pareto dominated by another profile: } \]
\[ \forall s \in S, s \neq s^*, u_i(s^*) > u_i(s) \text{ for all } i \in N. \]
\[ s^* \text{ is inefficient if it is dominated by some other profile.} \]
Cooperation and conflict: the iterated prisoner’s dilemma (R. Axelrod, 1984)

Why do players will play (Defet, Defet), as Nash Equilibrium proposes, if both prisioners could reduce the sentence by playing (Not Defet, Not Defet) ?

**Key:** (Not Defet, Not Defet) is appropiate from a social standpoint (as a group of players), while (Defet, Defet) is from the individual point of view.

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<thead>
<tr>
<th>1\2</th>
<th>Not Defet</th>
<th>Defet</th>
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<tbody>
<tr>
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<tr>
<td>Defet</td>
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Which is the predictable result if the two prisoners play it in practice?

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<tr>
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</tbody>
</table>

- If the game is played only once and both prisoners know that they are not able to communicate before deciding its strategy and they are rational: (Defet, Defet)

- However, **if the game is played several times, the above argument loses force**:
  - If the **game** is **played exactly N times** and both players know this, then it is **always** game theoretically **optimal to defect in all rounds**. The only possible NE is to always defect.
    - The proof is inductive: one might as well defect on the last turn, since the opponent will not have a chance to punish the player. Therefore, both will defect on the last turn. Thus, the player might as well defect on the second-to-last turn, since the opponent will defect on the last no matter what is done, and so on. The same applies if the game length is unknown but has a known upper limit.
  - **For cooperation to emerge** between game theoretic rational players, the **total number of rounds N must be random, or at least unknown to the players**. In this case always defect may no longer be a strictly dominant strategy, only a Nash equilibrium. Amongst results shown by R. Aumann in a 1959 paper, rational players repeatedly interacting for indefinitely long games can sustain the cooperative outcome.
Mixed strategies in finite games

• The set of strategies in a finite set are not convex sets, then Nash theorem cannot be applied to them.

• A finite game without NE: the matching euros.

• There is a “theoretical trick” that allows us to extend the game and to guarantee the existence of NE of the extended version of every finite game:
  
  • Consists of enlarging the strategic possibilities of the players and allowing them to choose not only the strategies they initially had (henceforth called pure strategies), but also the lotteries over their (finite) sets of pure strategies.

  • This extension of the original game is called its mixed extension, and the strategies are called mixed strategies.
A **mixed strategy** is an assignment of a probability to each pure strategy (a player can randomly select a pure strategy). There are infinitely many mixed strategies available to a player, even if their strategy set is finite.

- A **pure strategy** can be regarded as a degenerate case of a mixed strategy, in which that particular pure strategy is selected with probability 1 and every other strategy with probability.
- A **totally mixed strategy** is a mixed strategy in which the player assigns a strictly positive probability to every pure strategy.

\[
S_i = \{s_1^i, s_2^i, \ldots, s_k^i\} \text{ pure strategy set for player } i (= 1, 2, \ldots, n)
\]

Mixed strategy for player \(i\) : \(\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^k)\) lottery over \(S_i\)

\[
\delta(S_i) = \left\{\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^k) \mid \sigma_i^j \geq 0, \sum_{j=1}^{k} \sigma_i^j = 1\right\}
\]

A pure strategy is a mixed strategy : \(s_i^j = (0, \ldots, 1, \ldots, 0)\), \(j = 1, \ldots, k\)

Every convex linear combination of pure strategies is a mixed strategy

Expected payoffs : \(U_i (\sigma_1, \sigma_2, \ldots, \sigma_n) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \ldots \sum_{s_n \in S_n} \sigma_1^1 \sigma_2^2 \ldots \sigma_n^k u_i (s_1^1, s_2^2, \ldots, s_n^k)\)
Mixed strategies for matching euros

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<tr>
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<td>1</td>
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<tr>
<td>X</td>
<td>-1, 1</td>
<td>1, -1</td>
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\[
S_1 - S_2 = \{C, X\}
\]

\[
\chi(S_1) = \{(p, 1 - p) / 0 \leq p \leq 1\}; \text{ p prob. of choosing } C, (1 - p) \text{ prob. of } X
\]

\[
\chi(S_2) = \{(q, 1 - q) / 0 \leq q \leq 1\};
\]

\[
U_1((p, 1 - p), C) = pu_1(C, C) + (1 - p)u_1(X, C) = 2p - 1;
\]

\[
U_2((p, 1 - p), C) = pu_2(C, C) + (1 - p)u_2(X, C) = 1 - 2p.
\]

\[
U_1((p, 1 - p), X) = pu_1(C, X) + (1 - p)u_1(X, X) = 1 - 2p;
\]

\[
U_2((p, 1 - p), X) = pu_2(C, X) + (1 - p)u_2(X, X) = 2p - 1;
\]

\[
\begin{align*}
U_1((p, 1 - p), (q, 1 - q)) &= qU_1((p, 1 - p), C) + (1 - q)U_1((p, 1 - p), X) = 1 - 2p - 2q + 4pq; \\
U_2((p, 1 - p), (q, 1 - q)) &= pq(-1) + (1 - p)q(1) + p(1 - q)(1) + (1 - p)(1 - q)(-1) = 1 + 2p + 2q - 4pq.
\end{align*}
\]

\[
U_1((1 / 3, 2 / 3), C) = -1 / 3; U_2((1 / 3, 2 / 3), C) = 1 / 3.
\]

\[
U_1((1 / 3, 2 / 3), (4 / 5, 1 / 5)) = -3 / 15; U_2((1 / 3, 2 / 3), (4 / 5, 1 / 5)) = 3 / 15.
\]
Mixed strategies Nash equilibrium in finite games

• The mixed extension of a finite game satisfies the conditions of the Nash theorem: it always has, at least, one Nash equilibrium (statement proved by Nah in his original paper).

\[ G = \{N; S_1, \ldots, S_n; u_1, u_2, \ldots, u_n\} \]

\[ \sigma^* = (\sigma^* 1, \ldots, \sigma^* i, \ldots, \sigma^* n) \] is mixed strategy NE if for every player \( i \),

\[ U_i(\sigma^* 1, \ldots, \sigma^* i-1, \sigma^* i, \sigma^* i+1, \ldots, \sigma^* n) \geq U_i(\sigma^* 1, \sigma^* 2, \ldots, \sigma^* i-1, \sigma^* i, \sigma^* i+1, \ldots, \sigma^* n) \] for all \( \sigma^*_i \in \Delta(S_i) \).

\[ \sigma^* i \in \Delta(S_i) \] is a best response to \( \sigma^* -i \in \Delta(S_{-i}) \).

**Theorem (Nash, 1950).** Every finite strategic game \( G=\{N; S_1, \ldots, S_n; u_1, \ldots, u_n\} \) has, at least, one Nash equilibrium.
2.1.2. Mixed strategies NE in finite games

Characterization of mixed strategies NE in finite games

• Mixed strategies NE assign a strictly positive probability only those pure strategies that are best response to the strategies of other players.

**Theorem.** Let $G=\{N; S_1, ..., S_n; u_1, ..., u_n\}$ a finite strategic game. Then, the mixed strategy profile $\sigma^*=(\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ is a NE if and only if for each player $i$ with mixed strategy $\sigma_i^*=(\sigma_i^1, \sigma_i^2, ..., \sigma_i^n)$ to be $\sigma_i^j > 0$ implies that the pure strategy $s_i^j$ is a best response to $\sigma_i^*=($ $\sigma_1^*, ..., \sigma_{i-1}^*, \sigma_{i+1}^*, ..., \sigma_n^*)$.

• **Consequence:** a mixed strategy is best response to given best response strategies (pure or mixed), only if so are its pure strategies with strictly positive probability. Hence, such pure strategies generate the same payoffs (optimal payoffs).
2.1.2. Computing mixed strategies NE for matching euros

Mixed strategies NE for matching euros

\[
\begin{array}{|c|c|c|}
\hline
1 \backslash 2 & C & X \\
\hline
C & 1, -1 & -1, 1 \\
X & -1, 1 & 1, -1 \\
\hline
\end{array}
\]

\[
\sigma^* = \left( \left( \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{1}{2} \right) \right)
\]

\[
p = \frac{u_2(X, X) - u_2(X, C)}{u_2(C, C) + u_2(X, X) - u_2(X, C) - u_2(C, X)};
\]

\[
q = \frac{u_1(X, X) - u_1(C, X)}{u_1(C, C) + u_1(X, X) - u_1(X, C) - u_1(C, X)}.
\]
Computing mixed strategy NE in finite strategic games

**Games 2x2:** games with 2 players and 2 strategies. Using the above formula.

**Bimatrix games:** games with 2 players and a finite number of strategies. They can be easily represented using matrix notation \((A_{l\times m}, B_{l\times m})\), where \(A\) is the payoff matrix for player 1 and \(B\) the payoff matrix for player 2.

- Matrix games: \((A_{l\times m}, -A_{l\times m})\)
- Symetric games: \((A_{m\times m}, A^t_{m\times m})\)
- The algorithm by [Rahul Savani](http://banach.lse.ac.uk/) enumerates all equilibria of a bimatrix game (http://banach.lse.ac.uk/).

**Finite games:** games with a finite number of players and strategies.

- Gambit is a library of game theory software and tools for the construction and analysis of finite extensive and strategic games (http://www.gambit-project.org/doc/index.html).
Dynamic games with complete information

• **Elements:** players, nodes (nature, decision, terminal), possible actions (branches), information sets, probability assignment, and payoff for each player.

• **“Games with sequential moves”:** one player chooses his action before the others choose theirs
  - Importantly, the later players must have some information of the first's choice, otherwise the difference in time would have no strategic effect.
  - They are governed by the time axis, and represented in the form of decision trees.

• Again, it is **common knowledge** the complete structure of the game.

• **Extensive Games representation**

\[
\Gamma = \{N, (X, \sigma), (A, \alpha), \{X_i\}_{i \in N}, \{H_i\}_{i \in N}, (A(h))_{h \in H}, \rho, \pi\}
\]
Classes of dynamic games

- **With perfect information**: every information set, for each player, contains exactly one node.
  - The players know the full history of the game whenever it is their time to move: they know all the moves made by the other players and by the nature.
  - Chess, *Auction of a 50€ note*.

- **With imperfect information**: there exists, at least, a player with a no single information set
  - Cards, matching euros, *prisoner’s dilemma*.

- **With perfect recall**: each player, in each of his information sets, remembers what he has known and done in all his previous information sets.
  - Every game with perfect information is a game with perfect recall.

- Selten (1975) argues that games with imperfect recall are not a natural way to model noncooperative situations with completely rational players.
The iterated prisoner’s dilemma (imperfect information)

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<td>-2, -2</td>
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</table>

Two players play prisoners' dilemma more than once in succession and they remember previous actions of their opponent and change their strategy accordingly. We assume that the game is played exactly $n$ times, that both players know this, and the final payoffs are the sum of the payoffs corresponding to the different iterations.
The prisoner’s dilemma iterated 2 times
Auction of a 50 euros note (perfect information)

Pure strategies for player 1
- : P, P if J1 20 and J2 40
- : P, 60 if J1 20 and J2 40
- : 20, P if J1 20 and J2 40
- : 20, 60 if J1 20 and J2 40

Pure strategies for player 2
- : P if J1 P and P si J1 20
- : P if J1 P and 40 si J1 20
- : 20 if J1 P and P si J1 20
- : 20 if J1 P and 40 si J1 20

\[\text{NE}(G)= \{(s_1^4, s_2^4), (s_1^4, s_2^3), (s_1^3, s_2^4), (s_1^2, s_2^4)\}\]
2.2.2. Examples: auction of a 50 euros note

Auction of a 50 euros note (perfect information)

Pure strategies for player 1
- : P, P if J1 20 and J2 40
- : P, 60 if J1 20 and J2 40
- : 20, P if J1 20 and J2 40
- : 20, 60 if J1 20 and J2 40

Pure strategies for player 2
- : P if J1 P and P si J1 20
- : P if J1 P and 40 si J1 20
- : 20 if J1 P and J1 20
- : 20 if J1 P and 40 si J1 20

NE based on NO credible threat : (s₁⁴, s₂¹)

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<tr>
<th>Blanca \ Carlos</th>
<th>s₂¹</th>
<th>s₂²</th>
<th>s₂³</th>
<th>s₂⁴</th>
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<tbody>
<tr>
<td>s₁¹</td>
<td>25, 25</td>
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<td>s₁²</td>
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<tr>
<td>s₁³</td>
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<td>-20, 10</td>
<td>30, 0</td>
<td>-20, 10</td>
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<tr>
<td>s₁⁴</td>
<td>30, 0</td>
<td>-10, -40</td>
<td>30, 0</td>
<td>-10, -40</td>
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**Auction of a 50 euros note (perfect information)**

Pure strategies for player 1
- : P, P if J1 20 and J2 40
- : P, 60 if J1 20 and J2 40
- : 20, P if J1 20 and J2 40
- : 20, 60 if J1 20 and J2 40

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- : 20 if J1 P and 40 si J1 20

**NE based on NO credible threat :** (s₁¹, s₂⁴)

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<tr>
<td>s₁³</td>
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Auction of a 50 euros note (perfect information)

Pure strategies for player 1
• : P, P if J1 20 and J2 40
• : P, 60 if J1 20 and J2 40
• : 20, P if J1 20 and J2 40
• : 20, 60 if J1 20 and J2 40

Pure strategies for player 2
• : P if J1 P and P si J1 20
• : P if J1 P and 40 si J1 20
• : 20 if J1 P and P si J1 20
• : 20 if J1 P and 40 si J1 20

NE based on NO credible threat : (s^1_2, s^4_2)

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<tr>
<th>Blanca\Carlos</th>
<th>S^1_2</th>
<th>S^2_2</th>
<th>S^3_2</th>
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<tr>
<td>S^2_1</td>
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</tr>
<tr>
<td>S^4_1</td>
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<td>30, 0</td>
<td>-10, -40</td>
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Auction of a 50 euros note (perfect information)

Pure strategies for player 1
- P, P if J1 20 and J2 40
- P, 60 if J1 20 and J2 40
- 20, P if J1 20 and J2 40
- 20, 60 if J1 20 and J2 40

Pure strategies for player 2
- P if J1 P and P si J1 20
- P if J1 P and 40 si J1 20
- 20 if J1 P and P si J1 20
- 20 if J1 P and 40 si J1 20

NE based on CREDIBLE threat: \((s_1^4, s_2^3)\)

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<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
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<tr>
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<td>-20, 10</td>
</tr>
<tr>
<td>(s_4)</td>
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<td>30,0</td>
<td>-10, -40</td>
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</table>
2.2.2. Examples: auction of a 50 euros note

Auction of a 50 euros note (perfect information)

Pure strategies for player 1
- : P, P if J1 20 and J2 40
- : P, 60 if J1 20 and J2 40
- : 20, P if J1 20 and J2 40
- : 20, 60 if J1 20 and J2 40

Pure strategies for player 2
- : P if J1 P and P si J1 20
- : P if J1 P and 40 si J1 20
- : 20 if J1 P and P si J1 20
- : 20 if J1 P and 40 si J1 20

NE(G) = \{(s_1^4, s_2^1), (s_1^4, s_2^3), (s_1^1, s_2^4), (s_1^2, s_2^4)\}
SPNE(G) = \{(s_1^4, s_2^3)\}

<table>
<thead>
<tr>
<th>Blanca</th>
<th>Carlos</th>
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<tbody>
<tr>
<td>( s_2^1 )</td>
<td>( s_2^2 )</td>
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<td>( s_1^2 )</td>
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<td>( s_1^3 )</td>
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<tr>
<td>( s_1^4 )</td>
<td>30, 0</td>
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Strategies in finite dynamic games: mixed strategies v.s. behavior strategies

**• Behavior strategies:** a behavior strategy of player i is a map that assigns, to each choice when he is in at an information set, a probability

- Every pure strategy of a player can be seen as a behavior strategy.

**•** Given an extensive game, we can define the strategic game associated with it. If the latter is finite, we can consider its mixed extension. Then, a mixed strategy for a finite extensive game is a lottery over the pure strategies of it (i.e., a mixed strategy of the mixed extension of the corresponding strategic game).

**•** What is the relationship between the mixed strategies and the behavior strategies of an extensive game?

- Every behavior strategy induces a lottery over pure strategies in the natural way: for each behavior strategy (BS), there is an equivalent (i.e., induces the same probabilities over the set of pure strategies) mixed strategy (MS).

- Unfortunately, the converse is not true (as the next example shows).
2.2.3. Strategies in finite dynamic games: an example

Player 1: \{R_1, L_1\}

Player 2: \{(L_2, l_2), (L_2, r_2), (R_2, l_2), (R_2, r_2)\}

Nash Equilibria: \{\(L_1, (L_2, l_2)\), \(L_1, [(3/4, 1/4), (1, 0)]\)\}

Pure strategy: \(L_1, (L_2, l_2)\)

Behavior strategy: \(L_1, [(3/4, 1/4), (1, 0)]\)  \(\rightarrow\) mixed strategy: \((1, 0), (3/4, 0, 1/4, 0)\)

Mixed strategy for player 2: \((1/2, 0, 0, 1/2)\). There is no behavior strategy equivalent to it. When using BS, the players have to independently select the probabilities at the different information sets, whereas with MS the different choices can be correlated with one another (both sets are different even if we identify equivalent strategies).
The following results are consequences of Kuhn Theorem

Kuhn Theorem says that whatever a player can get with a mixed strategy, in a game with perfect recall, can also be achieved by a realization equivalent behavior strategy.

**Theorem.** Let \( \Gamma \) be an extensive game with perfect recall. Then, \( \Gamma \) has, at least, one Nash equilibrium.

**Theorem.** Let \( \Gamma \) be an extensive game with perfect recall. Every NE of \( \Gamma \) is also a NE of the mixed extension of the corresponding strategic game.
2.2.3. Backward Induction: SPNE

**Backward Induction: Subgame Perfect Nash Equilibrium (Selten, 1965)**

- The **Nash equilibrium (NE)** may also have non-rational consequences in sequential games (extensive games) because
  - players may “threaten” each other with non-rational moves;
  - might be **based on irrational plans in some information sets** that are not reachable if this equilibrium is played (see **auctions of 50 euros note**).

- The “literal” translation of the definition of NE concept to extensive games does not take into account the dynamic aspects of these games.

- Selten (1975) argued that, if commitments are not possible, then the behavior in a subgame can depend only on the subgame itself. Hence, only those NE of the original game that induce NE in all the subgames are sensible: **subgame perfect Nash equilibrium (SPNE)** may be more meaningful as a tool of analysis.

- **SPNE:** NE based on credible treats by players (credibility that can be given to future decisions determining the present decisions).
2.2.3. Backward Induction: SPNE

Subgames (Selten, 1965)

A subgame is a part of the game that can reasonably be considered as a game in themselves.

• Given a dynamic game with complete information G, and a decision node x of G, we say that **G' is a subgame of G starting in x** if G' is a part of G that satisfies the following conditions:
  a) G’ contains the node x and all nodes the follow, and only to them,
  b) Node x i a singleton information set,
  c) If a node y belongs to G’, also belong to G’ all nodes in the information set that belongs to; i.e., G’ does not break any information set.

**NOTES:**
• G is always a subgame of itself. A **proper subgame** is any subgame different from G.
• If G is a perfect information game, any part starting in a decision node and containing all the follower nodes is a subgame.
• A subgame can start at a random node, since it can be considered as a singleton information set.
• All subgames with a singler player are also subgames.
• Static games represented as extensive games have no proper subgames: the initial node is the only singleton information set.
Subgames for the auction of 50 euros note

2.2.3. Backward Induction: SPNE

![Diagram of subgames for the auction of 50 euros note]
2.2.3. Backward Induction: SPNE

Subgames for Prisoner’s dilemma iterated 2 times

Diagram showing the strategic decision models for the Prisoner's dilemma with iterated backward induction.
Subgame Perfect Nash Equilibrium (Selten, 1965)

Let G be an extensive game

a) Given s a NE of G, we say that s is SPNE of G if the restriction of s to any subgame of G is a NE of such a subgame.

b) Given r a result or possible development of G, we say that r is a subgame perfect result (SPR) of G if r can be obtained as an embodiment of a SPNE.

Theorem (existency). Every finite dynamic game G has, at least, a SPNE.

NOTE: It may happen that there is no SPNE in pure strategies, implying that some or all strategies that such a SPNE consists of are positive mixed strategies (assign a positive probability to each pure strategy).
2.2.3. Backward Induction: SPNE

Subgames for the auction of 50 euros note

SPNE(G) = \{(s_1^4, s_2^3)\}

s_1^4: 20, 60 if J1 20 and J2 40
s_2^3: 20 if J1 P and 40 si J1 20
Subgames for the matching euros game (imperfect information)

- There is no proper subgame, all NE are SPNE (the only subgame is the initial game).

- There is a unique NE, and so a unique SPNE: the strategy profile \((1/2, 1/2), (1/2, 1/2)\)
Summary 2

1. **No binding commitments** are possible in non cooperative models. Such a models are represented by **non cooperative games** which deals with strategies and payoffs, and considers players willing to use strategies that maximize their individual payoffs.

2. There are **four kinds of non cooperative games**: static and dynamic (extensive) games with complete (deterministic games) or incomplete (bayesian games) information.

3. **Static games with complete information are games with simultaneous moves and are represented as** strategic games. Each player knows all the information about the game.

4. **Nash Equilibrium** (NE) for a static game with complete information is a strategy profile such that no player gains when unilaterally deviating from it.

1. The **mixed extension** of the original game consists of enlarging the strategic possibilities of the players and allowing them to choose not only the strategies they initially had (pure strategies), but also the lotteries over their (finite) sets of pure strategies. The strategies here are called **mixed strategies**.

5. It may happen that there is no NE in pure strategies, but **every finite strategic game has, at least, one NE** in mixed strategies (Nash, 1950).

6. **Dynamic games with complete information are games with sequential moves and are represented as extensive games.** Each player knows all the information about the game but one player chooses his action before the others choose theirs.

7. **Repeated games (games iterated more than once in succession) are dynamic games.**
Summary 2

8. We can consider three classes of dynamic games: with perfect (every information set, for each player, contains exactly one node) or imperfect (there exists, at least, a player with a no single information set) information, and with perfect recall (each player, in each of his information sets, remembers what he has known and done in all his previous information sets). Every game with perfect information is a game with perfect recall.

9. A behavior strategy (BS) of player i is a map that assigns, to each choice when he is in at an information set, a probability. A mixed strategy (MS) for a finite extensive game is a lottery over the pure strategies of it. Every behavior strategy induces a lottery over pure strategies in the natural way: for each behavior strategy, there is an equivalent mixed strategy, but the converse is not true.

10. A subgame is a part of the game that can reasonably be considered as a game in themselves.

11. Subgame perfect Nash equilibrium (SPNE), NE based on credible treats by players, may be more meaningful as a tool of analysis. SPNE are those NE of the original game that induce NE in all the subgames (Selten, 1965).

12. A SPNE in pure strategies is always obtained by backward induction (the process of working from the end of a game back to the beginning in order to solve it).

13. It may happen that there is no SPNE in pure strategies, but every finite dynamic game has, at least, one SPNE in mixed strategies (Selten, 1965).

14. Gambit is a library of game theory software and tools for the construction and analysis of finite extensive and strategic games (http://www.gambit-project.org/doc/index.html).
Key Terms 2

- No binding commitments
- Non cooperative games
- *Static games*
- Games with complete information
- Games with incomplete information
- *Pure strategy*
- Nash Equilibrium
- *Mixed extension*
- Mixed strategy
- Dynamic games

- *Games with perfect information*
- *Games with imperfect information*
- *Games with perfect recall*
- Repeated games
- Behavior strategy
- Subgame
- Subgame perfect Nash equilibrium
- Backward induction
- Gambit Library
Problems 2

2.1. Give and analyze a general formulation for the following 2-person games: (1) Prisoner’s dilemma, (2) battle of the sexes, (3) Hawk-Dove.

2.2. Calculate all the NE for the following games: (1) producers of goods, (2) Prisoner’s dilemma, (3) battle of the sexes, and (4) Hawk-Dove.

2.3. Consider the following multistage interactive situation, with two players and three stages. Represent the strategic game associated with this multistage situation. Solve both of the games and compare them.

2.4. Consider the matching euros game. Suppose that the players, besides choosing C or X, can choose a lottery L that selects C with probability \( \frac{1}{2} \) and X with probability \( \frac{1}{2} \) (think, for instance, of a coin toss). Represent the new strategic game and solve it. Compare this solution with the one obtained for the original matching euros game.

2.5. Calculate and interpret the unique PSNE for the Prisoner’s dilemma iterated 2 times. There exists any other NE for this game?

2.6. Check that the unique SPNE for the card game is \( ((1/3, 2/3,0,0), (2/3,1/3)) \).

2.7. Describe and study the quantity competition model between two firms (Cournot Duopoly, 1838). Generalize this study to the case of more than two firms (Cournot Oligopoly, 1838).

2.8. Formulate and analyze the price competition model between two firms (Bertrand Duopoly, 1878).

2.9. Present and solve the credible quantity competition model (Stackelberg leadership model, 1934).
3. COOPERATIVE MODELS


3.2. Set solutions: the imputation set and the core (Gillies, 1953). Bondareva-Shapley conditions for the non-emptiness of the core. Examples

3.3. The core for convex/concave games. Examples

3.4. Point solutions: proportional allocations (Moulin, 1988), the Shapley value (Shapley, 1953), the Nucleolus (Schmeidler, 1969). Examples.

Practical sessions

3.5. The glove market (Aumann, 1987): the core
3.6. The airport game (Littlechild and Owen, 1973): the core and the Shapley value
3.8. The bankruptcy game (O’Niell, 1982): the Nucleolus vs. the Shapley value
Cooperative models (von Neumann y Morgenstern, 1947)

There are no restrictions on the agreements that may be reached among the players: there will be a tendency for players, whose objectives in the game are close, to form alliances or coalitions, and allocate the joint benefits derived from their cooperation (however it takes place). The structure given to the game by coalition formation is conveniently studied by reducing the game to a form in which coalitions play a central role.

- **With Transferible Utility (TU):** all transfers of utility across players are assumed to be possible
  - Implicitly assumes that there is a *numéraire* good (for instance, money) such that the utilities of all players are linear with respect to it and that this good can be freely transferred among players.

- **Without Transferible Utility (NTU):** utility is NOT transferable among players
  - Even if money is available in large amounts (which is not always the case), we find that the several players’ utility for money is not linear, and not always independent of their other assets.

We focus on **TU-games** (a number for each coalition of players) that are much more tractable than general NTU-games (a set for each coalition).
The banker game (Owen, 1972)

\[ v(\{i\}) = \{x_i / x_i \geq 0\}, i = 1,2,3 \]
\[ v(\{1,2\}) = \{(x_1, x_2) / x_1 + 4x_2 \leq 1.000, x_1 \leq 1.000\} \]
\[ v(\{1,3\}) = \{(x_1, x_3) / x_1 \leq 0, x_3 \leq 0\} \]
\[ v(\{2,3\}) = \{(x_2, x_3) / x_2 \leq 0, x_3 \leq 0\} \]
\[ v(N) = \{(x_1, x_2, x_3) / x_1 + x_2 + x_3 \leq 1.000\} \]

One can think of this game in the following way. On its own, no player can get anything. Player 1, with the help of player 2, can get 1,000 dollars. Player 1 can reward player 2 by sending him money, but the money sent is lost or stolen with probability 0.75. Player 3 is a banker, so player 1 can ensure his transactions are safely delivered to player 2 by using player 3 as an intermediary.

The question is how much should player 1 pay to player 2 for his help to get the 1,000 dollars and how much to player 3 for helping him to make costless transactions to player 2.

Some transfers among the players may not be allowed: (1,000, 0) belongs to \(v(\{1,2\})\), but players 1 and 2 cannot agree to share (500, 500) without the help of player 3.
Cooperative Games

Let \( n \geq 2 \) denote the number of players in the game, numbered from 1 to \( n \), and let \( N \) denote the set of players, \( N = \{1, 2, \ldots, n\} \). A coalition, \( S \), is defined to be a subset of \( N \), \( S \subset N \), and the set of all coalitions is denoted by \( 2^N \). By convention, we also speak of the empty set, \( \emptyset \), as a coalition, the empty coalition. The set \( N \) is also a coalition, called the grand coalition.

The coalitional form of an \( n \)-person game is given by the pair \( (N, v) \), where \( N = \{1, 2, \ldots, n\} \) is the set of players and \( v \) is a real-valued function, called the characteristic function of the game, defined on the set, \( 2^N \), of all coalitions, and satisfying \( v(\emptyset) = 0 \).

Compared to the strategic or extensive forms of \( n \)-person games, this is a very simple definition. Naturally, much detail is lost. The quantity \( v(S) \) is a real number for each coalition \( S \subset N \), which may be considered as the value, or worth, or power, of coalition \( S \) when its members act together as a unit.
Classes of TU cooperative games

- **0-normalize games**: \( v(\{i\}) = 0 \), for all \( i \in N \)

- **0-1-normalize games**: \( v(N) = 1, v(\{i\}) = 0 \), for all \( i \in N \).

- **Monotonic games**: \( v(S) \leq v(T) \); \( c(S) \geq c(T) \), for all \( S \subseteq T \subseteq N \)

- **Convex (concave) games** (Shapley, 1971): \( v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \) \( (c(T \cup \{i\}) - c(T) \leq c(S \cup \{i\}) - c(S)) \) for all \( i \in N \) and every \( S \subseteq T \subseteq N \setminus \{i\} \). It says that the larger a coalition is, the greater the marginal contribution value of each player to such as coalition is.

- **Superadditive (subadditive) games**: \( v(S) + v(T) \leq v(S \cup T) \) \( (c(S \cup T) \leq c(S) + c(T)) \), for all \( S, T \subseteq N \) s.t. \( S \cap T = \emptyset \). It says that the value of two disjoint coalitions is at least as great when they work together as when they work apart.

Convexity (concavity) \( \Rightarrow \) Superadditivities (subadditivities)
Set solutions: the imputation set and the core

- In superadditive (or convex) cooperative games, it is to the joint benefit of the players to form the grand coalition, \( N \), since the amount received, \( v(N) \), is as large as the total amount received by any disjoint set of coalitions they could form.

- As in the study of 2-person TU games, it is reasonable to suppose that “rational” players will agree to form the grand coalition and receive \( v(N) \).

- The problem is then to agree on how this amount should be split among the players

  - We discuss one of the possible properties of an agreement on a fair division, that it be stable in the sense that no coalition should have the desire and power to upset the agreement.
  - Such divisions of the total return are called points of the core, a central notion of game theory in economics.
The imputation set

- A payoff vector \( x = (x_1, x_2, \ldots, x_n) \) of proposed amounts to be received by the players, with the understanding that player \( i \) is going to receive \( x_i \) is called an allocation.

- The first desirable property of an allocation is that the total amount received by the players should be \( v(N) \): a payoff vector, \( x = (x_1, x_2, \ldots, x_n) \), is said to be group rational or efficient if \( \sum_{i=1}^{n} x_i = v(N) \).

- No player could be expected to agree to receive less than that player could obtain acting alone: an allocation, \( x \), is said to be individually rational if \( x_i \geq v(\{i\}) \) for all \( i = 1, \ldots, n \).

- Imputations are defined to be those allocations that satisfy both these conditions: an imputation is a payoff vector that is group rational and individually rational. The imputation set may be written

\[
I(v) = \left\{ x = (x_1, \ldots, x_n) / \sum_{i=1}^{n} x_i = v(N), x_i \geq v(i) \text{ for all } i \in N \right\}.
\]
3. 2. Set solutions: the imputation set

The imputation set

- The imputation set is never empty for games $v$ with $v(N) \geq \sum_{i=1}^{n} v(i)$. By the contrary, it is empty when $v(N) < \sum_{i=1}^{n} v(i)$.

- For example, one imputation is given by $x = (x_1, \ldots, x_n) / x_i = v(i)$ for all $i = 1, \ldots, n-1$, and $x_n = v(N) - \sum_{i=1}^{n-1} x_i$. This is the imputation most preferred by player $n$.

- In fact the set of imputations is exactly the simplex consisting of the convex hull of the $n$ points obtained by letting $x_i = v({i})$ for all $x_i$ except one, which is then chosen to satisfy $v(N) = \sum_{i=1}^{n} v(i)$.

Example 1: $v(1) = \frac{1}{2}$, $v(2) = 0$, $v(3) = \frac{3}{4}$, $v(N) = 9$, $v(12) = 3$, $v(13) = 5/2$, and $v(23) = 2$

$$I(v) = \{(x_1, x_2, x_3) / x_1 + x_2 + x_3 = 9, x_i \geq 1/2, x_2 \geq 0, x_3 \geq 5/2 \}.$$

This is an (equilateral) triangle each of whose vertices satisfy two of the three inequalities with equality: $(33/4, 0, 3/4)$, $(1/2, 31/4, 3/4)$, and $(1/2, 0, 17/2)$. These are the imputations most preferred by players 1, 2, and 3 respectively.
The imputation set for essential games

There is one trivial case in which the set of imputations consists of one point. Such a game is called inessential.

A cooperative TU game is said to be inessential if \( v(N) = \sum_{i=1}^{n} v(i) \), and essential if \( v(N) > \sum_{i=1}^{n} v(i) \).

If a game is inessential, then the unique imputation is \( x = (v(1), \ldots, v(n)) \), which may be considered the “solution” of the game. Every player can expect to receive his safety level.

From the game-theoretic viewpoint, inessential games are trivial. For every coalition \( S \), \( v(S) \) is determined by \( v(S) = \sum_{i \in S} v(i) \). There is no tendency for the players to form coalitions.

In Example 1, \( v(1) + v(2) + v(3) = 1/2 + 0 + 3/4 < 9 = v(N) \), so the game is essential, i.e, non-trivial. It is also a convex game.
The core (Gillies, 1953)

Suppose some imputation, \( x \), is being proposed as a division of \( v(N) \) among the players.

If there exists a coalition, \( S \), whose total return from \( x \) is less than what that coalition can achieve acting by itself, that is, if \( v(S) > \sum_{i \in S} x_i \) then there will be a tendency for coalition \( S \) to form and upset the proposed \( x \) because such a coalition could guarantee each of its members more than they would receive from \( x \). Such an imputation has an inherent instability.

An imputation \( x \) is said to be unstable through a coalition \( S \) if \( v(S) > \sum_{i \in S} x_i \).

We say \( x \) is unstable if there is a coalition \( S \) such that \( x \) is unstable through \( S \), and we say \( x \) is stable otherwise.

The set, \( C(v) \), of stable imputations of the game \( (N,v) \) is called the core.
The core (Gillies, 1953)

Formally,

\[
\mathcal{C}(v) = \left\{ x = (x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \right\}
\]

- The core can consist of either a single point or many points as in the examples below; but the core can also be empty (it may be impossible to satisfy all the coalitions at the same time).

- One may take the size of the core as a measure of stability, or of how likely it is that a negotiated agreement is prone to be upset.

- One class of games with empty cores are the essential constant-sum games:

\[
(N, v) / \text{for all } S \subseteq N, v(S) > \sum_{i \in S} x_i \text{ and } v(S) + v(N \setminus S) = v(N)
\]
Properties for the core

\[ C(v) = \left\{ x = (x_1, ..., x_n) \mid \sum_{i=1}^{n} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \right\} \]

(i) \( x \in C(v) \Rightarrow v(i) \leq x_i \leq v(N) - v(N \setminus i), \forall i \in N \)

(ii) \( C(v) \) is a polyhedron convex and compact in \( \mathbb{R}^n \)

(iii) If \( C(v) \neq \emptyset \), there always exist \( x^1, x^2, ..., x^n \in C(v) \), extreme points, such that \( C(v) = \text{conv}\{x^1, x^2, ..., x^n\} \)
Example 2

• \((N, \nu)\): \(\nu(1) = \nu(3) = 1, \nu(2) = 0, \nu(12) = 4, \nu(13) = 3, \nu(23) = 5\), and \(\nu(123) = 8\).

• It is \textit{not convex} since \(\nu(123) - \nu(23) = 3 < 4 = \nu(12) - \nu(2)\).

• The imputations are the points \((x_1, x_2, x_3)\) such that \(x_1 + x_2 + x_3 = 8\) and \(x_1 \geq 1, x_2 \geq 0, x_3 \geq 1\). This set is the triangle with vertices \((7, 0, 1), (1, 6, 1)\) and \((1, 0, 7)\).

• It is useful to plot this triangle in \textit{barycentric coordinates}

  • This is done by pretending that the plane of the plot is the plane \(x_1 + x_2 + x_3 = 8\), and giving each point on the plane three coordinates which add to 8.
  • Then it is easy to draw the lines \(x_1 = 1\) or the line \(x_1 + x_3 = 3\) (which is the same as the line \(x_2 = 5\)), etc.
  • It then becomes apparent that the set of imputations is an equilateral triangle.
Example 2

Let us find which imputations are unstable. The coalition \{2, 3\} can guarantee itself \(v(23) = 5\), so all points \((x_1, x_2, x_3)\) with \(x_2 + x_3 < 5\) are unstable through \{2, 3\}. These are the points below the line \(x_2 + x_3 = 5\) in the diagram. Since \{1, 2\} can guarantee itself \(v(12) = 4\), all points below and to the right of the line \(x_1 + x_3 = 4\) are unstable. Finally, since \{1, 3\} can guarantee itself \(v(13) = 3\), all points below the line \(x_1 + x_3 = 3\) are unstable.

The core is the remaining set of points in the set of imputations given by the 5-sided figure in the diagram, including the boundary: \(C(v) = \text{conv}\{\{1, 3, 4\}, (1, 5, 2), (2, 5, 1), (3, 4, 1), (3, 1, 4)\}\)
Example 3: Auction

A certain objet d’art is worth $a_i$ dollars to player $i=1, 2, 3$. Assume $a_1 < a_2 < a_3$, so player 3 values the object most. But player 1 owns this object so $v(1) = a_1$. Player 2 and 3 by themselves can do nothing, so $v(2) = 0$, $v(3) = 0$, and $v(2, 3) = 0$. If players 1 and 2 come together, the joint worth is $a_2$, so $v(1, 2) = a_2$. Similarly, $v(1, 3) = a_3$. If all three get together, the object is still only worth $a_3$, so $v(N) = a_3$.

The core consists of all vectors $(x_1, x_2, x_3)$ satisfying

$x_1 \geq a_1, x_2 \geq 0, x_3 \geq 0$

$x_1 + x_2 \geq a_2, \ x_1 + x_3 \geq a_3, x_2 + x_3 \geq 0$, and

$x_1 + x_2 + x_3 = a_3$.

$C(v) = \{(x, 0, a_3 - x)/ a_2 \leq x \leq a_3\}$.

This indicates that the object will be purchased by player 3 at some purchase price $x$ between $a_2$ and $a_3$. Player 1 ends up with $x$ dollars and player 3 ends up with the object minus $x$ dollars. Player 2 plays no active role in this, but without her around Player 3 might hope to get the object for less than $a_2$. 

Strategic Decision Models 2013
Example 4

• \((N,\nu)\): \(\nu(1) = \nu(3) = \nu(2) = 0, \nu(12) = \nu(13) = 60, \nu(23) = 80\), and \(\nu(123) = 100\).

• It is not convex since \(\nu(123) - \nu(23) = 20 < 60 = \nu(12) - \nu(2)\).

• The core shrinks to a single point: \(C(\nu) = \{(20, 40, 40)\}\). It consists of the vector of marginal contributions

\[
M(\nu) = (M_1(\nu), \ldots, M_n(\nu)) = (\nu(N) - \nu(N\backslash 1), \nu(N) - \nu(N\backslash 2), \nu(N) - \nu(N\backslash 3))
\]

• Let \((N,\nu)\) be a TU game such that \(C(N,\nu) \neq \emptyset\). If the vector of marginal contributions \(M(\nu)\) is efficient, then it is the unique point in the core of the game.

• Notice that the core of a game can be consists of a single point which is not the vector of marginal contributions:

\(N = \{1,2,3\}: \nu(1) = 1, \nu(2) = 2, \nu(3) = 3; \nu(12) = 2, \nu(13) = 4, \nu(23) = 5; \nu(N) = 6.\)

\(C(\nu) = I(\nu) = \{(1,2,3)\}.\)
Example 5: the rich dude

Four individuals (2,3,4,5) have a very rich and old dude (1) who are his only family. He wish to open a business either with one and only one of them or with all of them. If the business is opened they will get 100 million of euros of benefit. On the other hand, the four nephews can join them and undertake business with the money from the dude by a judicial declaration of inability dementia uncle, for which four of them have to declare it.

\[(N,v): v(12)=v(13)=v(14)=v(15)=v(2345)=v(N)=100, v(S)=0, \text{ for any other coalition } S.\]

\[
C(v)=\emptyset
\]

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 + x_5 &= 100 \\
    x_1 + x_2 &\geq 100 \\
    x_1 + x_3 &\geq 100 \\
    x_1 + x_4 &\geq 100 \\
    x_1 + x_5 &\geq 100 \\
    x_2 + x_3 + x_4 + x_5 &\geq 100 \\
    x_i &\geq 0, \forall i = 1,2,3,4,5
\end{align*}
\]
What conditions must satisfy the characteristic function of a game so that the core is not empty?

**A necessary condition:** If $C(N,v) \neq \emptyset$, then for all partition $S_1, S_2, \ldots, S_k$ of $N$,

$$v(S_1) + v(S_2) + \ldots + v(S_k) \leq v(N).$$

Those inequalities are not enough to ensure non-emptiness of the core. For instance, the simple game $v(i)=0$, $v(ij)=v(N)=1$, for all $i,j \in N$, satisfy the above inequalities but it has an empty core.

$C(N,v)=\emptyset$

$$x_1 + x_2 + x_3 = 1$$

$$v(1) + v(2) + v(3) = 0 \leq v(N) = 1,$$

$$x_1 + x_2 \geq 1$$

$$v(12) + v(3) = 1 = v(N),$$

$$x_1 + x_3 \geq 1$$

$$v(13) + v(2) = 1 = v(N),$$

$$x_2 + x_3 \geq 1$$

$$v(23) + v(1) = 1 = v(N).$$

$$x_i \geq 0, \forall i = 1,2,3$$
Bondareva-Shapley conditions [Bondareva (1963), Shapley (1967)]

Existence of core allocations for a game \((N,v)\) is equivalent to the linear problem (primal)

\[
\min \ x_1 + x_2 + \ldots + x_n \\
\text{s.a. } \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N, S \neq \emptyset
\]

has a solution, with optimal value \(v(N)\).

The dual problem is given by

\[
\max \ \sum_{\emptyset \neq S \subseteq N} v(S) y_S \\
\text{sujeto a } \sum_{i \in S} y_S = 1, \forall i = 1,\ldots,n \\
y_S \geq 0, \forall S \subseteq N, S \neq \emptyset
\]

It is well known by Duality Theory that if the primal problem has a solution, the dual also, and both optimal values are equal.

The condition for the dual has a solution is that the maximum value is at most \(v(N)\). This suggests the following conditions.
Bondareva-Shapley conditions

A collection \( C=\{S_1, S_2, \ldots, S_k\} \) of non-empty subsets of \( N \) is a balanced collection if there exist \( \alpha_{S_1}, \ldots, \alpha_{S_k} \in \mathbb{R}, \alpha_{S_j} > 0, j=1,\ldots,k / \sum_{\{i \in S_j\}} \alpha_{S_j} = 1, \forall i \in N \)

where \( \alpha_{S_j} \) are the weights corresponding to each coalition \( S_j \).

For \( n=3 \), some balanced collections are the following:

\[
\begin{align*}
  C_1 &= \{\{1\}, \{2\}, \{3\}\} \quad \text{con pesos } \alpha_1 = \alpha_2 = \alpha_3 = 1 \\
  C_2 &= \{\{1\}, \{2,3\}\} \quad \text{con pesos } \alpha_1 = \alpha_{32} = 1 \\
  C_3 &= \{\{2\}, \{1,3\}\} \quad \text{con pesos } \alpha_2 = \alpha_{13} = 1 \\
  C_4 &= \{\{3\}, \{1,2\}\} \quad \text{con pesos } \alpha_3 = \alpha_{12} = 1 \\
  C_5 &= \{\{1,2\}, \{1,3\}\} \quad \text{con pesos } \alpha_{12} = \alpha_{13} = \alpha_{23} = \frac{1}{2}.
\end{align*}
\]

A game \((N,v)\) is balanced if for every balanced collection \( C=\{S_1, S_2, \ldots, S_k\} \) of \( N \), with weights \( \alpha_{S_1}, \ldots, \alpha_{S_k} \) is satisfied \( \sum_{j=1}^{k} \alpha_{S_j} v(S_j) \leq v(N) \).

**Theorem (Bondareva/Shapley characterization).** \((N,v)\) is balanced if and only if \( C(N,v) \neq \emptyset \).
The core for convex/concave games (Shapley, 1971)

The extreme points for the core a convex/concave game are the vector of marginal contributions with respect to any possible order of the players (Shapley, 1971)

\[ (N, w), \sigma = (i_1, i_2, \ldots, i_n) \in \Pi(N) \]

\[ m^\sigma (N, w) \in \mathbb{R}^n \]

\[ m_{i_1}^\sigma (N, w) := w(\{i_1\}), \]

\[ m_{i_2}^\sigma (N, w) := w(\{i_1, i_2\}) - w(\{i_1\}), \]

\[ \ldots \]

\[ m_{i_n}^\sigma (N, w) := w(\{i_1, i_2, \ldots, i_n\}) - w(\{i_1, i_2, \ldots, i_{n-1}\}). \]

**Characterization of convex/concave games:** \((N,w)\) is convex (concave) if and only if \( C(N, w) = \text{conv} \{ m^\sigma (N, w)/ \sigma \in \Pi(N) \}. \)
### Example 1

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<table>
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<td>v(123)-v(12)</td>
</tr>
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<td>v(1)</td>
<td>v(123)-v(13)</td>
<td>v(13)-v(1)</td>
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<td>213</td>
<td>v(12)-v(2)</td>
<td>v(2)</td>
<td>v(123)-v(12)</td>
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<td>v(123)-v(13)</td>
<td>v(3)</td>
</tr>
<tr>
<td>321</td>
<td>v(123)-v(23)</td>
<td>v(23)-v(3)</td>
<td>v(3)</td>
</tr>
</tbody>
</table>

\[
I(v) = \text{conv } \{(0.5, 0, 8.5), (8.25, 0, 0.75), (0.5, 7.75, 0.75)\}
\]

\[
C(v) = \text{conv } \{(0.5, 2.5, 6), (0.5, 6.5, 2), (3, 0, 6), (7, 0, 2), (1.75, 6.5, 0.75), (7, 1.25, 0.75)\}
Example 1

\[ C(v) = \text{conv} \{(0.5, 2.5, 6), (0.5, 6.5, 2), (3, 0, 6), (7, 0, 2), (1.75, 6.5, 0.75), (7, 1.25, 0.75)\} \]
Point solutions

• The concept of the core is useful as a measure of stability. As a set solution concept, it presents a set of imputations without distinguishing one point of the set as preferable to another. Indeed, the core may be empty.

• Next we deal with the concept of a value. In this approach, one tries to assign to each game in coalitional form a unique vector of payoffs, called the value. The i\textsuperscript{th} entry of the value vector may be considered as a measure of the value or power of the i\textsuperscript{th} player in the game.

• Alternatively, the value vector may be thought of as an arbitration outcome of the game decided upon by some fair and impartial arbiter. The central “value concept” in game theory is the one proposed by Shapley in 1953.

• There are another value concepts as the proportional allocations introduced by Moulin en 1988, and the nucleolus showed in Schmeidler (1969).
Proportional allocations (Moulin, 1988)

Proportional allocation with respect to a parameter: \((N, w)\)

- It allocates the grand coalition value in a proportional way according to the fixed factor \(\lambda \in \mathbb{R}^n \setminus \{0_n\}\)

\[
p_i(N, w) := \sum_{j \in N} \frac{\lambda_j}{\lambda_i} w(N), \quad \forall i \in N,
\]

\[
p(N, w) = \frac{\lambda}{\lambda(N)} w(N), \quad \text{con} \quad \lambda(N) =: \sum_{j \in N} \lambda_j.
\]

Examples: proportional to individual costs
Core allocation
\(p(N, v) = (72, 36, 48) * 360/156 = (166.15, 83.07, 110.76)\)

\[
\begin{array}{c|ccccccccc}
S & \emptyset & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\
\hline
v(S) & 0 & 72 & 36 & 48 & 192 & 192 & 144 & 360 \\
\end{array}
\]

\(p(N, c) = (4, 6, 20) * 20/30 = (2.66, 4, 13.33)\)

No core allocation: \(p_1(c) + p_2(c) = 6.66 > c(\{1,2\}) = 6\).
The Shapley Axioms (Shapley, 1953)

- As an example of the type of reasoning involved in arbitrating a game, consider Example 2, \((N,v)\) s.t.

\[ v(1) = v(3) = 1, \ v(2) = 0, \ v(12) = 4, \ v(13) = 3, \ v(23) = 5, \text{ and } v(123) = 8. \]

- Certainly the arbiter should require the players to form the grand coalition to receive 8, but how should this be split among the players?

- Player 2 can get nothing by himself, yet he is more valuable than 1 or 3 in forming coalitions.

- Which is more important?

- We approach this problem by axiomatizing the concept of fairness.
The Shapley Axioms (Shapley, 1953)

A value function, $\varphi$, is a function that assigns to each possible characteristic function of an $n$-person game, $\nu$, an $n$-tuple, $\varphi(\nu) = (\varphi_1(\nu), \varphi_2(\nu), \ldots, \varphi_n(\nu))$ of real numbers. Here $\varphi_i(\nu)$ represents the worth or value of player $i$ in the game with characteristic function $\nu$. The axioms of fairness are placed on the function, $\varphi$.

The Shapley axioms for $\varphi(\nu)$:

1. Efficiency: $\sum_{i \in N} \varphi_i(N, \nu) = \nu(N)$

1. Symmetry: $\forall i, j \in N / \nu(S \cup \{i\}) = \nu(S \cup \{j\}), \forall S \subseteq N \setminus \{i, j\}$ symmetric players, $\varphi_i(N, w) = \varphi_j(N, w)$

1. Dummy player: $\forall i \in N / \nu(S \cup \{i\}) - \nu(S) = \nu(\{i\}), \forall S \subseteq N \setminus \{i\}$ dummy player, $\varphi_i(N, w) = \nu(\{i\})$

1. Additivity: $\varphi_i(N, \nu + w) = \varphi_i(N, \nu) + \varphi_i(N, w), \forall i \in N$
The Shapley Axioms (Shapley, 1953)

- Axiom 1 is group rationality, that the total value of the players is the value of the grand coalition.

- The second axiom says that if players $i$ and $j$ are symmetric in $v$, then the values assigned to $i$ and $j$ should be equal.

- The third axiom says that if player $i$ is a dummy in the sense that he neither helps nor harms any coalition he may join, then his value should be zero.

- The strongest axiom is number 4. It reflects the feeling that the arbitrated value of two games played at the same time should be the sum of the arbitrated values of the games if they are played at different times.

**Theorem (Shapley, 1953).** There existis a unique function $\phi$ satisfying the Shapley axioms.
An alternative form of the Shapley value (Shapley, 1953)

- Suppose we form the grand coalition by entering the players into this coalition one at a time.

- As each player enters the coalition, he receives the amount by which his entry increases the value of the coalition he enters.

- The amount a player receives by this scheme depends on the order in which the players are entered.

- The Shapley value is just the average payoff to the players if the players are entered in completely random order.

\[
\varphi_i(N, w) := \sum_{\substack{S \subseteq N \setminus \{i\} \subseteq N \cap \{i\} \subseteq S \subseteq N \setminus \{i\}} \frac{S!(n-S-1)!}{n!} [w(S) - w(S \setminus \{i\})], \forall i \in N;
\]

\[
\varphi(N, w) = (\varphi_i(N, w))_{i \in N}.
\]
An alternative form of the Shapley value: interpretation

\[ \phi_i(N, w) := \sum_{\substack{S \subseteq N \\text{and} \ i \in S}} \frac{s!(n - s - 1)!}{n!} \left[ w(S) - w(S \setminus \{i\}) \right], \forall i \in N. \]

Suppose we choose a random order of the players with all \( n! \) Orders (permutations) of the players equally likely. Then we enter the players according to this order. If, when player \( i \) is enters, he forms coalition \( S \) (that is, if he finds \( S - \{i\} \) there already), he receives the amount \([v(S) - v(S \setminus \{i\})]\).

The probability that when \( i \) enters he will find coalition \( S - \{i\} \) there already is \((|S| - 1)! (n - |S|)!/n!\). The denominator is the total number of permutations of the \( n \) players. The numerator is number of these permutations in which the \( S-1 \) members of \( S\setminus\{i\} \) come first \(((|S| - 1)! \text{ ways})\), then player \( i \), and then the remaining \( n - |S| \) players \((n - |S|)! \text{ ways})\). So this formula shows that \( \phi_i(v) \) is just the average amount player \( i \) contributes to the grand coalition if the players sequentially form this coalition in a random order.
Another alternative form of the Shapley value (Shapley, 1971)

*The Shapley value can be also obtained as the average of all vector of marginal contributions with respect to any possible order of the players.*

\[
\phi_i(N, w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, w), \forall i \in N;
\]

\[
\phi(N, w) = (\phi_i(N, w))_{i \in N}.
\]

- The Shapley value is always efficient.

- If \((N, w)\) is *superadditive (subadditive)*, the Shapley value is an *imputation* (individually rational).

- If \((N, w)\) is *convex (concave)*, the Shapley value is a core allocation (*stable*).
3.4 The Shapley value: examples

The Shapley value: examples

• Shapley is not individually rational

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<td>2</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>10</td>
</tr>
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</table>

φ(v)=(11/6,10/3,29/6)
No superadditive: there exist {1},{2} such that v(1)+v(2)=4>3=v(12)

• Shapley is not stable

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<tr>
<td>v(S)</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
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</table>

φ(v)=(3/2,2,5/2)
Imputation, but does not belong to the core: there exist {2,3} such that φ_2(v)+φ_3(v)=9/2<5=v(23)

No convex: v({1,2})-v({2})=3>1=v({1,2,3})-v({2,3})
The Shapley value: examples

- **Producers of goods game (convex):**

  \[
  \begin{array}{cccccccc}
  S & \emptyset & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\
  v(S) & 0 & 72 & 36 & 48 & 192 & 192 & 144 & 360 \\
  \end{array}
  \]

- \(\phi(v) = (146, 104, 110)\); Barycenter of the core

\[
\begin{array}{ccc}
\sigma \backslash \text{players} & 1 & 2 & 3 \\
123 & 72 & 120 & 168 \\
132 & 72 & 168 & 120 \\
213 & 156 & 36 & 168 \\
231 & 216 & 36 & 108 \\
312 & 144 & 168 & 48 \\
321 & 216 & 96 & 48 \\
\text{Total Shapley} & 876 & 624 & 660 \\
\end{array}
\]
Another characterization of the Shapley value (Young, 1985)

A new axiom is introduced by Young in 1985:

5. Monotonicity: \[ \forall i \in N / v(S \cup \{i\}) \leq w(S \cup \{i\}), \forall S \subseteq N \setminus \{i\}, \phi_i(N, v) \leq \phi_i(N, w) \]

If we change dummy player (Axiom 3) and additivity (Axiom 4) by monotonicity (Axiom 5), we obtain the Shapley value.

**Theorem (Young, 1985).** The Shapley value is the unique efficient allocation satisfying symmetry and monotonicity axioms.
Proportional allocations v.s. the Shapley value

• Neither the proportional rule nor the Shapley Value are core allocations, in general.

• The Shapley value is computationally complex

  • Core allocation for convex (concave) games (Shapley, 1971),
  • There are around 20 axiomatology characterizations (Shapley, 1953; Young, 1985; Hart and Mas-Colell, 1989, etc),

  • Littlechild and Owen en 1973 prove a simple and very intuitive formula for the Shapley value of the Airport game,
  • Fragnelli and Meca (2007) provide a simple expresión of the Shapley value for games corresponding to von Neumann-Morgenstern market situations (von Neumann and Morgenstern, 1947).

• Proportional rule is computationally easy and, moreover, is a core allocation for the class of p-additive games.
3. 4. Point solutions: the nucleoulus

The nucleoulus (Schmeidler, 1969)

Perhaps the second most important point solution for TU games, just behind the Shapley value.

The nucleoulus consists of those imputations the minimize the vector of nonincreasingly ordered excess according to the lexicographic order within the set of imputations (i.e., it is not defined for inessential games).

The excess of coalition S with respect to x: a measure of the degree of dissatisfaction of coalition S when the allocation x es realized.

\[(N, \nu), x \in \mathbb{R}^+, S \subseteq N \quad \forall x \in I(\nu), e(N, x) = 0.\]

\[e(S, x) := \nu(S) - \sum_{i \in S} x_i \quad x \in C(\nu) \Rightarrow e(S, x) \leq 0, \forall S \subseteq N.\]

The vector of ordered excesses \(\theta(x) \in \mathbb{R}^{2^N}\) is the vector whose components are the excesses of the coalitions in N arranged in nonincreasing order:

\[\theta(x) = (e(S, x))_{S \subseteq N} = (\theta_1(x), \theta_2(x), ..., \theta_{2^N}(x)) \text{ with } \theta_k(x) \geq \theta_{k+1}(x), \forall k = 1, 2, ..., 2^N - 1\]
Example 6. \((N, v), N = \{1,2,3\}, v(1) = v(2) = 0, v(3) = 1, v(12) = 2, v(13) = v(23) = 1, v(N) = 2 \)
\(x = (1,0,1), y = (3/2,1/2,0), z = (1/2,1/2,1)\)
\(\theta(x) = (1,0,0,0,0,-1,-1)\)
\(\theta(y) = (1,1/2,0,0,0,-1/2,-1/2,-3/2)\)
\(\theta(z) = (1,0,0,0,-1/2,-1/2,-1/2,-1/2)\)
\(\theta(y) \succ_L \theta(x) \succ_L \theta(z)\)

The most dissatisfied coalitions with respect to \(x\) and \(y\) are equally dissatisfied, but the second most dissatisfied coalition is more dissatisfied in \(y\) than it is in \(x\). The idea of the nucleolus is to lexicographically minimize the degree of dissatisfaction of the different coalitions. With this fairness idea in mind, \(x\) would be more desirable than \(y\).

<table>
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<th>Coalitions</th>
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<td>2</td>
<td>1</td>
<td>1</td>
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<td>0</td>
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<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(e(S,y))</td>
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<td>-1/2</td>
<td>1</td>
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<td>-1/2</td>
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</tr>
<tr>
<td>(e(S,z))</td>
<td>0</td>
<td>-1/2</td>
<td>-1/2</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>-1/2</td>
<td>0</td>
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</table>
Some properties for the nucleolus

• The nucleolus of a balanced game is a **core element**.

• If the core of a balanced game is a singleton, then the core coincides with the nucleolus.

• The nucleolus is a **symmetric** point solution.

• Given a **dummy player**, the nucleolus assigns its own individual value.

• Since the nucleolus satisfies the first three axioms of the Shapley value, it **does not satisfy linearity** axiom.
3.4. Point solutions: the nucleolus

The lexicographic order:

\[ x, y \in R^n, \theta(y) \succ_L \theta(x) \]
\[ \exists l \in \mathbb{N}, 1 \leq l \leq 2^n \forall k \in \mathbb{N}, k < l, \theta_k(y) = \theta_k(x) \text{ and } \theta_l(y) > \theta_l(x). \]
\[ \theta(y) \succeq_L \theta(x) \text{ if either } \theta(y) \succ_L \theta(x) \text{ or } \theta(y) = \theta(x). \]

The nucleolus:

\[ (N, v), I(v) \neq \emptyset \]
\[ \eta(v) := \{ x \in I(v) | \theta(y) \succeq_L \theta(x), \forall y \in I(v) \} \]

It is well known that the nucleolus is actually a singleton, i.e., it is never empty and it contains a unique allocation.

Therefore, with a slight abuse of language, we identify the unique element of the nucleolus with the nucleolus itself and refer to \( \eta(v) \) as the nucleolus of \( v \).
3. 4. Computing the nucleoulus (Maschler et al., 1979)

A procedure to compute the nucleoulus (Maschler et al., 1979)

Consider the following LP

\[
\begin{align*}
\text{Min} & \quad \sum_{i \in S} x_i + \alpha_i \\
\text{subject to} & \quad \sum_{i \in S} x_i + \alpha_i \geq v(S), \varnothing \neq S \subseteq N \\
& \quad x \in I_0 
\end{align*}
\]

It has at least one optimal solution \( \{\alpha_1\} \times I_1 \)

If this set is a singleton, then \( I_1 \) coincides with the nucleoulus. Otherwise, let \( \Sigma_1 \) be the collection of coalitions given by

\[
\Sigma_1 := \{S \subseteq N/ \sum_{i \in S} x_i + \bar{\alpha}_i = v(S), \forall x \in I_1\}.
\]

For \( k > 1 \), solve the LP

\[
\begin{align*}
\text{Min} & \quad \sum_{i \in S} x_i + \alpha_i \\
\text{subject to} & \quad \sum_{i \in S} x_i + \alpha_i \geq v(S), \varnothing \neq S \subset N, S \not\in \bigcup_{l<k} \Sigma_l \\
& \quad x \in I_{k-1}
\end{align*}
\]

where, for each \( l < k \), \( \Sigma_l \) is the collection of coalitions given by

\[
\Sigma_l := \{S \subseteq N/ \sum_{i \in S} x_i + \bar{\alpha}_i = v(S), \forall x \in I_l\}.
\]

This algorithm finishes once the optimal solution set is a singleton (in at most \( 2^n - 1 \) steps).
Example 6.

\[ (N, \nu), N = \{1,2,3\}, \nu(1) = \nu(2) = 0, \nu(3) = 1, \nu(12) = 2, \nu(13) = \nu(23) = 1, \nu(N) = 2 \]

\[
\begin{align*}
    \min \quad & \alpha_i \\
    \text{s.t.} \quad & x_i + \alpha_i \geq 0, i \in \{1,2\} \\
                    & x_3 + \alpha_i \geq 1 \\
                    & x_1 + x_2 + \alpha_i \geq 2 \\
                    & x_1 + x_3 + \alpha_i \geq 1 \\
                    & x_2 + x_3 + \alpha_i \geq 1 \\
                    & x_i \geq 0, i \in \{1,2\} \\
                    & x_3 \geq 1 \\
                    & x_1 + x_2 + x_3 = 2
\end{align*}
\]

\[
\begin{align*}
    \min \quad & \alpha_i \\
    \text{s.t.} \quad & x_i + \alpha_i \geq 0, i \in \{1,2\} \\
                    & x_3 + \alpha_i \geq 1 \\
                    & x_1 + x_3 + \alpha_i \geq 1 \\
                    & x_2 + x_3 + \alpha_i \geq 1 \\
                    & x_i \geq 0, i \in \{1,2\} \\
                    & x_3 = 1; \mathcal{I}_i = \{\{1,2\}\}.
\end{align*}
\]

\[
\begin{align*}
    \min \quad & \alpha_i \\
    \text{s.t.} \quad & x_i + \alpha_i \geq 0 \\
                    & x_2 + \alpha_i \geq 0 \\
                    & x_i \geq 0, i \in \{1,2\} \\
                    & x_1 + x_2 = 1 \\
                    & x_3 = 1 \\
                    & x_1 + x_2 = 1
\end{align*}
\]

\[
\begin{align*}
    \min \quad & \alpha_i \\
    \text{s.t.} \quad & x_i + \alpha_i \geq 0, i \in \{1,2\} \\
                    & x_3 + \alpha_i \geq 1 \\
                    & x_1 + x_3 + \alpha_i \geq 1 \\
                    & x_2 + x_3 + \alpha_i \geq 1 \\
                    & x_i \geq 0, i \in \{1,2\} \\
                    & x_3 = 1; \mathcal{I}_2 = \{\{3\}\}.
\end{align*}
\]

\[ \eta(\nu) = (1/2, 1/2, 1) \]
Nucleolus v.s. Shapley value and proportional allocations

- The nucleolus of a balanced game is always a core allocation while neither the proportional rule nor the Shapley Value are core allocations, in general.

- **The nucleolus** is computationaly more complex than Shapley and proportional.

- **The Shapley value** is computationaly complex than proportional
  - Core allocation for convex (concave) games (Shapley, 1971),
  - Littlechild and Owen en 1973 prove a simple and very intuitive formula for the Shapley value of the Airport game,

- **The nucleolus and the Shapley value** can easily be calculated with TUGLAB for 3-4 players TU games.

- **Proporcional rule** is computationaly easy and, moreover, is a core allocation for the class of p-additive games.
The glove market (Aumann, 1987)

Let \( N \) consist of two types of players, \( N = R \cup L \), where \( R \cap L = \emptyset \). Let the characteristic function be defined by \( \nu(S) = \min\{|S \cap R|, |S \cap L|\} \).

\((N, \nu)\) is called the glove market because of the following interpretation: each player of \( L \) owns a right-hand glove and each player of \( L \) owns a left-hand glove. If \( j \) members of \( R \) and \( k \) members of \( L \) form a coalition, they have \( \min\{j, k\} \) complete pairs of gloves, each being worth 1. Unmatched gloves are worth nothing.

- If \(|R| < |L|\), the core of \( \nu \) has a single point that assigns 1 to each member of \( R \) and 0 to each member of \( L \).
- If \(|L| < |R|\), the core of \( \nu \) has a single point that assigns 1 to each member of \( L \) and 0 to each member of \( R \).
- If \(|R| = |L|\), \( C(N, \nu) = \{(x_1, \ldots, x_r, y_1, \ldots, y_r) / x_i = \varepsilon, \forall i = 1, \ldots, |L|; y_j = 1 - \varepsilon, \forall j = 1, \ldots, |R|\} \).
The airport game (Littlechild and Owen, 1973)

Consider the following cost allocation problem. Building an airfield will benefit \( n \) airplanes (players). Airplane \( j \) requires an airfield that costs \( c_j \) to build, so to accommodate all the airplanes, the field will be built at a cost of \( \max_{1 \leq j \leq n} c_j \). How should this cost be split among the players?

Suppose all the costs are distinct and let \( c_1 < c_2 < \ldots < c_n \). Take the characteristic function of the game to be \( v(S) = -\max_{j \in S} c_j \).

It is always a convex game. Hence, the core is the convex hull of the vector of marginal contributions with respect to any possible order of the airplanes.

The Shapley value

\[
\phi_1(N, c) = \frac{c_1}{n}; \\
\phi_2(N, c) = \frac{c_1}{n} + \frac{c_2 - c_1}{n - 1}; \\
\phi_3(N, c) = \frac{c_1}{n} + \frac{c_2 - c_1}{n - 1} + \frac{c_3 - c_2}{n - 2}; \\
\vdotss \\
\phi_n(N, c) = \frac{c_1}{n} + \frac{c_2 - c_1}{n - 1} + \frac{c_3 - c_2}{n - 2} + \ldots + \frac{c_{n-1} - c_{n-2}}{2} + c_n - c_{n-1}.
\]
3.7. The bankruptcy game (O’Niell, 1982): the Nucleolus vs. the Shapley value

The bankruptcy game (O’Niell, 1982)

A small company goes bankrupt owing money to three creditors. The company owes
creditor A $10,000, creditor B $20,000, and creditor C $30,000. If the company has only
$36,000 to cover these debts, how should the money be divided among the creditors?

A pro rata split of the money would lead to the allocation of $6000 for A, $12,000 for B,
and $18,000 for C, denoted by \( x = (6, 12, 18) \) in thousands of dollars. We shall compare
this allocation with those suggested by the Shapley value and the nucleolus.

First, we must decide on a characteristic function to represent this game. Of course we
will have \( v(\emptyset) = 0 \) from the definition of characteristic function, and \( v(ABC) = 36 \)
measured in thousands of dollars. By himself, A is not guaranteed to receive anything
since the other two could receive the whole amount; thus we take \( v(A) = 0 \). Similarly,
\( v(B) = 0 \). Creditor C is assured of receiving at least $6000, since even if A and B receive
the total amount of their claim, namely $30,000, that will leave $36,000 - $30,000 =
$6000 for C. Thus we take \( v(C) = 6 \). Similarly, we find \( v(AB) = 6 \), \( v(AC) = 16 \), and \( v(BC) = 26 \).

\[
\rho(v) = (6, 12, 18); \eta(v) = (5, 10.5, 20.5); \varphi(v) = (6, 11, 19)
\]
The bankruptcy game (O’Niell, 1982)

It is interesting to see how the nucleolus and the Shapley value change in the bankrupt company example as the total remaining assets of the company change from $0 to $60,000, that is, as \( v(N) \) changes from 0 to 60. Consider the nucleolus. If \( v(N) \) is between 0 and 15, the nucleolus divides this amount equally among the players. For \( v(N) \) between 15 and 25, the nucleolus splits the excess above 15 equally between \( B \) and \( C \), while for \( v(N) \) between 25 and 35, all the excess above 25 goes to \( C \). For \( v(N) \) between 35 and 45, the excess above 35 is split between \( B \) and \( C \), and for \( v(N) \) between 45 and 60, the excess above 45 is divided equally among the three players.

<table>
<thead>
<tr>
<th>Nucleolus</th>
<th>Amount of ( v(N) ) between 0 and 15: share equally</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15 and 25: ( B ) and ( C ) share</td>
</tr>
<tr>
<td></td>
<td>25 and 35: ( C ) gets it all</td>
</tr>
<tr>
<td></td>
<td>35 and 45: ( B ) and ( C ) share</td>
</tr>
<tr>
<td></td>
<td>45 and 60: share equally</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shapley Value</th>
<th>Amount of ( v(N) ) between 0 and 10: share equally</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10 and 20: ( B ) and ( C ) share</td>
</tr>
<tr>
<td></td>
<td>20 and 40: ( C ) gets 2/3rds and ( A ) and ( B ) get 1/6th</td>
</tr>
<tr>
<td></td>
<td>40 and 50: ( B ) and ( C ) share</td>
</tr>
<tr>
<td></td>
<td>50 and 60: share equally</td>
</tr>
</tbody>
</table>

Note that at \( v(N) = 30 \), the nucleolus and the Shapley value coincide with the pro rata point. Compared to the pro rata point, both the Shapley value and the nucleolus favor the weaker players if \( v(N) \) is small, and favor the stronger players if \( v(N) \) is large, more so for the Shapley value than the nucleolus.
Summary 3

1. **Cooperative models study situations where there are no restrictions on the agreements that may be reached among the players:** there will be a tendency for players, whose objectives in the game are close, to form alliances or coalitions, and allocate the joint benefits derived from their cooperation (however it takes place).

2. We can consider two classes of cooperative models, those with **Transferible Utility (TU)** (all transfers of utility across players are assumed to be possible) and those where utility is NOT transferable among players (NTU). Here we focus on **TU-games**.

3. In superadditive (or convex) games, it is to the joint benefit of the players to form the grand coalition, \( N \), since the amount received, \( v(N) \), is as large as the total amount received by any disjoint set of coalitions they could form. Hence, it is reasonable to suppose that “**rational**” players **will agree to form the grand coalition** and receive \( v(N) \).

4. The main objective is then to agree on how this amount should be split among the players.

5. **The first desirable property of an allocation is** that satisfy group rational and individually rational; i.e., to be an imputation.

6. **The imputation set is never empty for essential games. However, an imputation can be unstable through a coalition** \( S \), and so be a unstable imputation.

7. **The set of stable imputations (no unstable) is called the core (Gillies, 1963).**

8. The core can consist of either a single point or many points as in the examples below; but the core can also be empty. One may take the size of the core as a measure of stability, or of how likely it is that a negotiated agreement is prone to be upset.
Summary 3

9. Bondareva (1963) and Shapley (1967) provided, simultaneously but independently, a characterization for the nonemptiness of the core. The showed that a game is balanced if and only if its core is non empty.

10. Shapley (1971) showed that the extreme points for the core a convex game are the vector of marginal contributions with respect to any possible order of the players. In addtion, the Shapley value is the barycenter of the core of a convex game.

11. Alternatively to the core, a set solution concept, the “value concept” is proposed as a point solution concept in cooperative game theory. The central value concepts in game theory are the Shapley value, proposed in Shapley (1953), the proportional allocations introduced by Moulin en 1988, and the nucleolus showed in Schmeidler (1969).

12. The nucleolus of a balanced game is always a core allocation while neither the proportional rule nor the Shapley Value are core allocations, in general.

14. The Shapley value is computationally more complex than the proportional allocations. The nucleolus is computationally more complex than Shapley and proportional.

15. TUGLAB is a library of game theory software and tools for the construction and analysis of finite cooperative TU games (http://www.tuglabweb.co.nr/).
Key Terms 3

- Binding commitments
- Cooperative games
- TU and NTU cooperative games
- Convex games
- Superadditive games
- Imputations
- Stable imputations
- Core allocations
- Bodareva/Shapley conditions
- *The Shapley value*
- Proportional allocations
- *The nucleolus*
- TUGLAB library
Problems 3

3.1. A game with characteristic function \( v \) is said to be **symmetric** if \( v(S) \) depends only on the number of elements of \( S \), say \( v(S) = f(|S|) \) for some function \( f \).

- (a) In a symmetric 3-player game with \( v(i) = 0 \), \( v(i,j) = a \) and \( v(123) = 3 \), for what values of \( a \) is the core non-empty?
- (b) In a symmetric 4-player game with \( v(i) = 0 \), \( v(i,j) = a \), \( v(ijk) = b \), and \( v(N) = 4 \), for what values of \( a \) and \( b \) is the core non-empty?
- (c) Generalize. Find necessary and sufficient conditions on the values of \( f(|S|) = v(S) \) for a symmetric game to have a non-empty core.

3.2. Find the core and the nucleolus for the Producers of goods game.

3.3. Find the Shapley value for the games given in examples 1, 2, 4, 5, and 6. Is the Shapley value en the core of any of these games?

3.4. Find the nucleolus for the games given in examples 1, 2, 4, and 5.

3.5. *(Market with one seller and two buyers)* Player 1 owns an art object of no intrinsic worth to him. Therefore he wishes to sell it. The object is worth $30 to player 2 and $40 to player 3. Set this up as a game in characteristic function form. Find the Shapley value and the nucleolus. Is the Shapley value in the core? (Refer to Example 3).

3.6. *(The Cattle Drive)* Rancher A has some cattle ready for market, and he foresees a profit of $1200 on the sale. But two other ranchers lie between his ranch and the market town. The owners of these ranches, B and C, can deny passage through their land or require payment of a suitable fee. The question is: What constitutes a suitable fee? The characteristic function may be taken to be: \( v(A) = v(B) = v(C) = v(BC) = 0 \) and \( v(AB) = v(AC) = v(ABC) = 1200 \).

- (a) Find the core, and note that it consists of one point. This point must then be the nucleolus. (Why?)
- (b) Find the Shapley value.
- (c) Which do you think is more suitable for settling the question of a fee, the nucleolus or the Shapley value, and why?
3. Problems

3.7. (An Assignment Game). Two house owners, A and B, are expecting to sell their houses to two potential buyers, C and D, each wanting to buy one house at most. Players A and B value their houses at 10 and 20 respectively, in some unspecified units. In the same units, Player C values A’s house at 14 and B’s house at 23, while Player D values A’s house at 18 and B’s house at 25.

(a) Determine a characteristic function for the game.
(b) Find the Shapley value.
(c) Find the nucleolus.

3.8. (Simple Games). A game \((N, v)\) is simple if for every coalition \(S \subset N\), either \(v(S) = 0\) or \(v(S) = 1\). In a simple game, a coalition \(S\) is said to be a winning coalition if \(v(S) = 1\) and a losing coalition if \(v(S) = 0\). So in a simple game every coalition is either winning or losing. It follows from superadditivity of \(v\) that in simple games every subset of a losing coalition is losing, and every superset of a winning coalition is winning. For simple games, the Shapley value simplifies because the difference \([v(S) − v(S\ \setminus \{i\})]\) is always zero or one. It is zero if \(v(S)\) and \(v(S\ \setminus \{i\})\) are both zero or both one, and it is one otherwise. Therefore we may remove \([v(S) − v(S\ \setminus \{i\})]\) from Shapley formula provided we sum only over those coalitions \(S\) that are winning with \(i\) and losing without \(i\). Formula (6) for the Shapley value (the Shapley-Shubik Index) becomes

\[
\phi_i(v) = \sum_{S \text{ winning or losing}} \frac{(n-s)!(s-1)!}{n!} v(S) v(S\ \setminus \{i\})
\]

In a simple game a player \(i\) is said to be a veto player if \(v(N\ \setminus i) = 0\).

(a) Show that the core is empty if there are no veto players.
(b) Show, conversely, that the core is not empty if there is at least one veto player.
(c) Characterize the core.

3.9. There is a large class of simple games called weighted voting games. These games are defined by a characteristic function of the form

\[
v(S) = \begin{cases} 
1 & \text{if } \sum_{i \in S} w_i > q \\
0 & \text{if } \sum_{i \in S} w_i \leq q
\end{cases}
\]
3. Problems

for some non-negative numbers $w_i$, called the **weights**, and some positive number $q$, called the **quota**. If $q = (1/2) i \in \mathbb{N} w_i$, this is called a **weighted majority game**. Consider the game with players 1, 2, 3, and 4, having 10, 20, 30, and 40 shares of stock respectively, in a corporation. Decisions require approval by a majority (more than 50%) of the shares. This is a weighted majority game with weights $w_1 = 10$, $w_2 = 20$, $w_3 = 30$ and $w_4 = 40$ and with quota $q = 50$. Find the Shapley value and the core of this game.

**3.10.** Consider an airport game with $n=3$, $c_1=4$, $c_2=6$, $c_3=20$. Find the core, the Shapley value and the nucleolus. Compare the Shapley value to the nucleolus.

**3.11.** Consider a globe market game with 4 players where $L=\{1,3\}$ and $R=\{2,4\}$. Find the core, the Shapley value and the nucleolus. Compare the Shapley value to the nucleolus.

**3.12.** Consider a globe market game with 5 players where $L=\{1,3,5\}$ and $R=\{2,4\}$. Find the core, the Shapley value and the nucleolus. Compare the Shapley value to the nucleolus. How could you tell before computing it that the nucleolus was not the same as the Shapley value?
4. BIFORM MODELS: GAME THEORY APPLIED TO BUSINESS STRATEGY

4.1. An hybrid non cooperative-cooperative model: the biform game model

4.2. How to analyze business-strategy ideas using a biform game: Branded Ingredient Game

4.2.1. How will be the Branded Ingredient Game be played?

4.3. General formulation of a biform game

4.4. General analysis of the efficiency and inefficiency of business strategies

Practical sessions (Brandenburger and Stuart, 2007)

4.5. A Negative-Advertising Game
4.6. A Coordination Game
4.7. A Repositioning Game
4.1. An hybrid non cooperative-cooperative model: the biform game model

The biform game model (Brandenburger and Stuart, 2007)

- There have been a number of applications of game theory to the field of business strategy in recent years
  - Noncooperative model useful for analyzing strategic moves in business
  - Cooperative model useful for addressing the basic question of how much power the different players have in a given setting

- Since both models have a role to play in understanding business strategy, they put them together to create a hybrid non cooperative-cooperative model that they call the biform game model.

- A biform game is a two-stage game:
  - The first stage is noncooperative and is designed to describe the strategic moves of the players. However, the consequences of these moves are not payoffs (at least not directly).
  - Instead, each profile of strategic choices at the first stage leads to a second-stage cooperative game. This gives the competitive enviroment created by the choices that the players made in the first stage.
  - Analysis of the second stage then tells us how much value each player will capture (i.e., gives us the payoff of the players).
The biform game model (Brandenburger and Stuart, 2007)

- The biform model is precisely a formalization of the idea that business strategies shape the competitive environment and thereby the fortunes of the players.

- Gluing together the different formalisms gives a meaning to strategy as “choosing the game”.
A Branded Ingredient Game

• Two firms, each able to produce a single unit of a certain product.

• One supplier that can supply the necessary input to at most one firm, at a cost of $1.

• Numerous buyers, each interested in buying a single unit of the product from one of the two firms. Every buyer has a willingness-to-pay $9 for Firm A’s product and $3 for Firm B’s product.

• The supplier has the option of incurring an upfront cost of $1 to increase the buyers’ willingness-to-pay for Firm B’s product to $7.

• This is the branded-ingredient strategy, under which the supplier puts its logo on its customers’ products (we assume that this does not affect the buyers’ willingness-to-pay for Firm A, the stronger firm).
4.2.1. Branded Ingredient Game

Branded Ingredient Game

Supplier

Status quo

Branded-ingredient strategy

Firm A

$1

$9

$1

Firm B

$3

$1

$7
How will the Branded Ingredient Game be played?

• We start by analyzing each of the cooperative games (the endpoints of the tree induced by the supplier’s choices), and then work back to find the optimal strategy for the supplier.

• We will analyze cooperative games using the core to calculate the effect of competition among the players at the second stage of the game—i.e., given the strategic choices made in the first stage. This determines how much value each player can capture.

  • The core might be a single point. If so, competition fully determines the division of value.
  • However, there may also be a range of values in the core, so that competition alone is not fully determinate, and (at least some) players face a “residual” bargaining problem.

• The cores in the current example are as follows.
How will the Branded Ingredient Game be played?

In the **status quo game**, a total of $9-$1=$8 of value will be created. Firm B and the buyers will not capture any value.

The supplier will receive between $2(=3-1)$ and $8(=9-1)$, and Firm A will receive between $0(=8-8)$ and $6(=8-2)$ (where the sum of what the supplier and Firm A get must be $8$).

This is, of course, the intuitive answer: Competition among buyers ensures that the firms can get their full willingness-to-pay. Firm B can then bid up to $3$ for the input from the supplier. Firm A has an advantage over Firm 2 (because it commands the higher willingness-to-pay), and so will be the one to secure the input.

However, because of the presence of Firm B, it will have to pay at least $3$ for it. Thus, the supplier gets a minimum of $3-$1=$2 of value, and the remaining $9 - 3 = 6$ is subject to negotiation between the supplier and Firm A, and this could be split in any way.

\[
N = \{S, A, B\} \\
v(N) = 8, v(S) = v(A) = v(B) = 0, \\
v(SA) = 8, v(SB) = 2, v(AB) = 0, \\
C(v) = conv\{(8,0,0), (2,6,0)\}.
\]
How will the Branded Ingredient Game be played?

The analysis of the branded-ingredient game is very similar: $8 of value will be created gross of the $1 upfront cost, or $7 net. Again, Firm B and the buyers will not capture any value.

This time, the supplier is guaranteed $5, and the remaining $2 of value will be split somehow between the supplier and Firm A. We see that paying $1 to play the branded-ingredient strategy may well be worthwhile for the supplier. For example, if the supplier is cautious about how the “residual pies” will get divided between it and Firm A, then it will prefer the guaranteed $5 in the bottom game to the guaranteed $2 in the top game.

The “aha” of the strategy is that it would not be worthwhile for Firm B to pay $1 to increase willingness-to-pay for its product from $3 to $7. It would still be at a competitive disadvantage. However, it is worthwhile for a supplier (at least if cautious) to pay the $1 to increase this willingness-to-pay and thereby level the playing field. It gains by creating more equal competition among the firms. This may be at least one effect of a branded-ingredient strategy in practice.

\[
N = \{S, A, B\} \\
v(N) = 8 - 1 = 7, v(S) = v(A) = v(B) = 0, \\
v(SA) = 8 - 1 = 7, v(SB) = 6 - 1 = 5, v(AB) = 0, \\
C(v) = \text{conv}\{(7,0,0),(5,2,0)\}.
\]
Definition of a biform game

An n-player biform game is a collection

\[(S^1, ..., S^n; \nu; \alpha^1, ..., \alpha^n)\]

where

(a) for each \( i = 1, ..., n, S^i \) is a finite set
(b) \( \nu \) is a map from \( S := S^1 \times ... \times S^n \) to the set of maps from \( P(N) \) to the real numbers, with \( \nu(s)(\emptyset) = 0 \) for every \( s = (s^1, ..., s^n) \in S \)
(c) for each \( i = 1, ..., n, 0 \leq \alpha^i \leq 1 \).

The number \( \alpha^i \) is **player i’s confidence index**: indicates how well player i anticipates doing in the resulting cooperative games.

Note that the strategies sets \( S^1, ..., S^n \) can come from a general extensive form game, so the definition above is certainly not restricted to a normal form game.
Properties of a biform game

• A biform game will be called essential (respectively, superaditive) if the if the cooperative game $v(s)$ is essential (respectively, superadditive) for each $s \in S$.

• The biform model $(S^1,\ldots,S^n;\alpha^1,\ldots,\alpha^n)$ is a strict generalization of both the strategic-form noncooperative and TU cooperative game models. Remain that a $n$-player strategic-form noncooperative game is a collection $(S^1,\ldots,S^n;\pi^1,\ldots,\pi^n)$ where the sets $S^i$ are as above and, for each $i$, player $i$'s payoff function $\pi^i$ maps $S$ to the reals.

• Two remarks: fix confidence indices $\alpha^1,\ldots,\alpha^n$

  • There is a natural bijection between the subclass of $n$-player biform games that are essential and superadditive, and the class of $n$-player strategic-form noncooperative games.

  • There is a natural bijection between the subclass of $n$-player biform games which the sets $S^i$ are singletons, and the class of $n$-player TU cooperative games.
A procedure to analyze of a biform game

1. For every profile \( s \in S \) of strategic choices and resulting cooperative game \( v(s) \),
   1. compute the core of \( v(s) \),
   2. for each player \( i = 1,\ldots,n \), calculate the projection of the core onto the \( i \)th coordinate axis (which is a closed bounded interval in the real line),
   3. calculate the \( \alpha_i : (1 - \alpha_i) \) weighted average of the upper and lower endpoints of the projection.
2. For every profile \( s \in S \) of strategic choices, and each player \( i = 1,\ldots,n \),
   1. assign to player \( i \) a payoff equal to \( i \)'s weighted average as in (3) above,
   2. analyze the resulting strategic-form noncooperative game.

\[
s \in S \rightarrow v(s) \rightarrow C(v(s))
\]
\[
x \in C(v(s))
\]
For every \( i \in N \),
\[
v(s)(i) \leq x_i \leq v(s)(N) - v(s)(N | i),
\]
\[
\pi^i(s) = (1 - \alpha^i)v(s)(i) + \alpha^i[v(s)(N) - v(s)(N | i)].
\]

Use of the confidence indices reduces a biform game to a strategic-form noncooperative game (Step 2.1). This game may now be analyzed in standard fashion—say, by computing Nash equilibria, iteratively eliminating dominated strategies, or some other method (Step 2.2).
### Branded Ingredient Game-Payoff Ranges

In the status quo game, the range for the supplier was \([\$2, \$8]\); in the branded-ingredient game it was \([\$5, \$7]\). The supplier’s decision problem was then a choice between ranges of payoffs rather than single payoffs (see next Figure).

![Branded-Ingredient Game-Payoff Ranges Diagram]

In general, under some basic axioms, the players’ preferences over such intervals can be represented by confidence indices. An optimistic player \(i\) will have a confidence index \(\alpha^i\) close to one, indicating that player \(i\) anticipates capturing most of the value to be divided in the residual bargaining. A pessimistic player \(i\) will have an \(\alpha^i\) close to zero, indicating that player \(i\) anticipates getting little of this residual value. Thus, a player’s confidence index represents a view of the game—optimistic if the confidence index is large, and pessimistic if the confidence index is small.

Here, the supplier would choose the branded-ingredient strategy if \(7\alpha + (1 - \alpha) 5 > 8 \alpha + (1- \alpha)2\), or \(\alpha < \frac{3}{4}\), so the supplier would choose this strategy unless it was very optimistic. Use of the confidence indices gives us an induced noncooperative game among the players.
General analysis of the efficiency and inefficiency of business strategies

A strategy profile \((s^1, \ldots, s^n)\) is efficient (inefficient) if it leads to (does not lead to) the largest total value \([\text{solves } \max_{r \in S} v(r)(N)]\). It is always exists because we are dealing with finite sets \(S^i\).

Conditions of Biform Games \((S^1, \ldots, S^n; v; \alpha^1, \ldots, \alpha^n)\)

- **Adding Up (AU):** For each strategy profile \(s \in S\),
  \[
  \sum_{i=1}^n (v(s)(N) - v(s)(N \setminus \{i\})) = v(s)(N).
  \]

- **No Externalities (NE):**
  
  For each player \(i \in N\), pair of strategies \(r^i, s^i\) for \(i\), and strategy profile \(s^{-i}\) for the players other than \(i\),
  
  \(v(r^i, s^{-i})(N \setminus \{i\}) = v(s^i, s^{-i})(N \setminus \{i\})\).

- **No Coordination (NC):**
  
  For each player \(i \in N\), pair of strategies \(r^i, s^i\) for \(i\), and pair of strategy profile \(r^{-i}, s^{-i}\) for the players other than \(i\),
  
  \(v(r^i, r^{-i})(N) > v(s^i, r^{-i})(N)\) if and only if \(v(r^i, s^{-i})(N) > v(s^i, s^{-i})(N)\).
Main results in biform games

1. **Characterization of Nash equilibrium in biform games.** Fix a biform game satisfying AU, NE (and non-emptiness of the Core for each strategy profile). Then, a profile $s$ is a (pure) Nash equilibrium if and only if

$$v(s)(N) > v(r^i, s^{-i})(N) \text{ for every } r^i \in S^i.$$

2. Fix a biform game satisfying AU, NE, and NC (and non-emptiness of the Core for each strategy profile). Then, *if a profile $s$ is a (pure) Nash equilibrium, it is efficient.*

3. Fix a biform game satisfying AU and NE (and non-emptiness of the Core for each strategy profile). Then, *if a profile $s$ is efficient, it is a (pure) Nash equilibrium.*
4.4. General analysis of the efficiency and inefficiency: example

A Branded Ingredient Game

In the status-quo second-stage game, the supplier has an added value of $8, and Firm A has an added value of $6 (Firm B and the buyers have zero added value). The overall value is $8, so AU fails. (We could equally have looked at the branded-ingredient second-stage game.) NE holds: Only the supplier has a strategic choice, and the value created without the supplier is constant—at $0—regardless of which strategy the supplier follows. Finally, it is easy to see that NC is automatically satisfied in any game where only one player has a strategic choice.

Conclusion: the possible inefficiency in this game (choosing BI strategy with overall value 7 instead of SQ with total value 8) comes from the failure of AU. In plain terms, there is a bargaining problem between the supplier and Firm A over the $6 of value that must be divided between them. To do better in this bargaining, the supplier may well adopt a strategy (the branded-ingredient strategy) that decreases the pie.

\[
N = \{S, A, B\} \\
v(N) = 8, v(S) = v(A) = v(B) = 0, \\
v(SA) = 8, v(SB) = 2, v(AB) = 0, \\
C(v) = \text{conv}\{(8,0,0), (2,6,0)\}.
\]

\[
N = \{S, A, B\} \\
v(N) = 8 - 1 = 7, v(S) = v(A) = v(B) = 0, \\
v(SA) = 8 - 1 = 7, v(SB) = 6 - 1 = 5, v(AB) = 0, \\
C(v) = \text{conv}\{(7,0,0), (5,2,0)\}.
\]
A Negative-Advertising Game

• There are three firms, each with one unit to sell at $0 cost. There are two buyers, each interested in one unit of product from some firm.

• Firm A alone has a strategic choice, which is whether or not to engage in negative advertising

  • If it does not, then each buyer has a willingness-to-pay of $2 for each firm’s product.

  • If it does, then willingness-to-pay for its own product is unchanged, but that for Firm B’s and Firm C’s products falls to $1. (The negative advertising has hurt the image of Firms B and C in the eyes of the buyers.)
A Negative-Advertising Game

Status quo

Firm A

$0

Firm A

$2

Firm B

$0

Firm C

$2

Firm A

$2

Firm B

$1

Firm C

$1

$0

$0

$0

$0
A Negative-Advertising Game

• Along the status-quo branch, the overall value is $4 (=2*2). Each firm’s added value is $0 (=4-4); each buyer’s added value is $2 (=4-2).

• By efficiency of the marginal contribution vector, each firm will get $0 in the core, and each buyer will get $2. (This is the intuitive answer because the firms are identical, and supply exceeds demand.)

• Along the negative-advertising branch, the overall value is $3 (=2*1+1). Firm A’s added value is $1 (=3-2), Firm B’s and Firm C’s added values are $0 (=3-3), and each buyer’s added value is $1 (=2-1). Again using efficiency of the marginal contribution vector, Firms B and C will get $0 in the core, and Firm A and each of the buyers will get $1. (The core says that what the three firms offer in common is competed away to the buyers, while what Firm A uniquely offers is competed away to it by the two buyers.)

• Firm A will optimally choose the negative-advertising example, thereby capturing $1 versus $0. Note that the strategy is inefficient: it shrinks the overall value created from $4 to $3.

• AU is satisfied. NC is satisfied for the same reason as in the branded-ingredient game (only Firm A has a strategic choice). However, NE fails: the value created by Firm B, Firm C, and the two buyers is $4 when Firm A chooses the status-quo strategy, but changes to $2 when Firm A chooses the negative-advertising strategy. Again, the outcome is inefficient.
A Coordination Game

• There are three players, each with two strategies, labelled No and Yes.

• Player 1 chooses the row, Player 2 chooses the column, and Player 3 chooses the matrix. Next Figure depicts the cooperative game associated with each strategy profile, where the value of all one-player subsets is taken to be zero.

• This example can be thought of as a model of switching from an existing technology standard (the strategy No) to a new standard (the strategy Yes). The new technology costs $1 more per player, and is worth $2 more per player, provided at least two players adopt it.

\[
\begin{array}{|c|c|c|c|c|}
\hline
& \text{No} & \text{Yes} & & \\
\hline
\text{No} & \nu(\text{N}) = 6 & \nu(\text{N}) = 5 & \nu(1, 2) = 4 & \nu(1, 2) = 3 \\
& \nu(2, 3) = 4 & \nu(2, 3) = 3 & \nu(3, 1) = 4 & \nu(3, 1) = 3 \\
\text{Yes} & \nu(\text{N}) = 5 & \nu(\text{N}) = 6 & \nu(1, 2) = 3 & \nu(1, 2) = 6 \\
& \nu(2, 3) = 3 & \nu(2, 3) = 6 & \nu(3, 1) = 3 & \nu(3, 1) = 3 \\
\hline
\end{array}
\]
A Coordination Game

- Using the efficiency of the marginal vector, it is easy to check that in each of the four cooperative games, the core will give the players exactly their added values.
- We get the induced noncooperative game in next Figure, which is a kind of coordination game.

- There are two (pure-strategy) Nash equilibria: (No, No, No) and (Yes, Yes, Yes).
- The first is inefficient (the total value is $6), while the second is efficient (the total value is $9).

- AU is satisfied. So is NE, however, NC fails: for example, when Player 1 changes strategy from No to Yes, the overall value falls from $6 to $5 when Players 2 and 3 are both playing No, but rises from $6 to $9 when Players 2 and 3 are both playing Yes.
- There is an inefficient Nash equilibrium (No, No, No). Note that the efficient profile (Yes, Yes, Yes) is also a Nash equilibrium, as result 3 says it must be.

\[
\begin{array}{c|cc}
\text{No} & \text{Yes} \\ 
\hline 
\text{No} & 2, 2, 2 & 2, 1, 2 \\ 
\text{Yes} & 1, 2, 2 & 3, 3, 0 \\ 
\end{array}
\]

\[
\begin{array}{c|cc}
\text{No} & \text{Yes} \\ 
\hline 
\text{No} & 2, 2, 1 & 0, 3, 3 \\ 
\text{Yes} & 3, 0, 3 & 3, 3, 3 \\ 
\end{array}
\]
A Repositioning Game

• There are three firms, each with one unit to sell. There are two identical buyers, each interested in one unit of product from some firm.

• Under the status quo, the firms have the costs, and the buyers have the willingness-to-pay numbers, on the upper branch.

• Firm 2 has the possibility of repositioning, as shown by its vertical bar on the lower branch. Specifically, it can spend $1 to raise “quality” (willingness-to-pay) and lower cost as shown.
A Repositioning Game

• Along the status-quo branch, the overall value is $14 (=7+7). Each firm’s added value is $0 (=7-7); each buyer’s added value is $7 (=14-7).

• Along the repositioning branch, the overall value is $15 ($16 (=9+7) minus the $1 repositioning cost). Firm 2’s added value is $1; Firm 1’s and Firm 3’s added values are $0; each buyer’s added value is $7.

• AU is satisfied, as clearly are NE and NC. By the efficiency of the marginal contribution vector, we know that Firm 2 must optimally make the efficient choice of the lower branch, as indeed it will, to net $1.

• This is a simple biform model of a (re)positioning strategy. Note that on the lower branch, Firm 2 still does not have either the lowest cost or the highest quality among the three firms. However, it does command the largest gap between quality and cost. This is the source of its added value.
Some further comments on Results 2 and 3

• First, note that the Table 1 establishes the independence of the AU, NE, and NC conditions. It is possible for any two to hold, but not the third.

<table>
<thead>
<tr>
<th>Conditions Satisfied in Examples</th>
<th>Adding up</th>
<th>No externalities</th>
<th>No coordination</th>
</tr>
</thead>
<tbody>
<tr>
<td>Branded-ingredient game</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Negative-advertising game</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>Coordination game</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
</tbody>
</table>

• Next, we emphasize that the efficiency conditions in Result 2 are sufficient, not necessary (see Problem 4.1), where the (pure) Nash equilibria is efficient, but AU and NC fail (NE holds).

• This said, the conditions do seem “tight.” To maintain sufficiency, none of them can be left out in Result 2, as our examples showed.
Summary 4

1. The biform game model is an hybrid non cooperative-cooperative model. This is designed to formalize the notion of business strategy as making moves to try to shape the competitive environment in a favorable way.

2. A biform game is a two-stage game, with the first stage being a noncooperative game, and the second being a cooperative game.

3. The noncooperative game at the first stage is designed to describe the strategic moves of the players. The consequences of these moves are not payoffs, instead, each profile of strategic choices at the first stage leads to a second-stage cooperative game. This gives the competitive environment created by the choices that the players made in the first stage.

4. The second-stage cooperative game is analyzed by using the core to calculate the effect of competition among the players at the second stage of the game, given the strategic choices made in the first stage. This determines how much value each player can capture.

5. We show that if a biform model of a business strategy satisfies our three conditions, AU,NE and NC, then we will get efficiency.

6. We now have a general way of doing a kind of “audit” of any business strategy—if modelled as a biform game—to discover its efficiency properties.
4. Key terms

Key Terms 4

Biform games
Hybrid noncooperative-cooperative
Two-stage games
Strategic moves
Choosing the game
Confidence index
Game Payoff-Ranges

Efficency
Inefficiency
Core
Nash Equilibrium
Adding Up (AU)
No Externalities (NE)
No Coordination (NC)
Business-strategy
Problems 4

4.1. (Innovation Game) Consider the following game of innovation between two firms. Each firm has a capacity of two units, and (for simplicity) zero unit cost. There are three buyers, each interested in one unit of product. A buyer has a willingness-to-pay of $4 for the current-generation product, and $7 for the new-generation product. Each firm has to decide whether to spend $5 to bring out the new product. The biform game is depicted in the following Figure. (Each vertical bar represents one unit. As for the Branded Ingredient Game, upfront costs are not shown.)

![Biform Game Diagram]

4.2. (Market Feature not Market Failure) Consider two firms labeled F1 and F2, and two “employee-consumers” labeled E1 and E2. F1 decides whether or not to hire E1 to make its product. F2 decides whether or not to hire E2 to make its product. The cost of hiring an employee is c. If E1 is paid c, then he has a willingness-to-pay of w for F2’s product. If E2 is paid c, then she has a willingness-to-pay of w for F1’s product. Model and analyze the feature via the ‘crossover’ between 1 and 2 by using biform games.
Recommended Bibliography

Additional Bibliography